CONVERGENCE THEOREMS FOR ABSTRACT ASYMPTOTIC MARTINGALES

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In this dissertation convergence theorems are derived for asymptotic martingales which take values in a locally convex topological vector space and for scalar-valued asymptotic martingales which are integrable with respect to a measure which takes values in such a space. In chapter one we establish basic concepts and terminology concerning asymptotic martingales of real-valued random variables. In chapter two several convergence theorems are developed for weakly measurable Pettis integrable asymptotic martingales which take values in a locally convex topological vector space. We obtain conditions which insure a discrete-parametered asymptotic martingale converges almost everywhere in the weak topology as well as conditions guaranteeing weak convergence in measure of a continuous-parametered asymptotic martingale. In chapter three similar theorems are obtained for asymptotic martingales consisting of strongly measurable Pettis integrable random variables. In chapter four, a theory of asymptotic martingales is developed for scalar-valued random variables which are integrable with respect to a measure which takes values in a locally convex topological vector space. Properties of asymptotic martingales are investigated and an optional sampling theorem as well as several convergence theorems are proved.

INTRODUCTION

In this dissertation convergence theorems for asymptotic martingales are investigated in several settings. In particular, convergence theorems are derived for asymptotic martingales which take values in a locally convex linear topological vector space, and for scalar-valued asymptotic martingales which are integrable with respect to a measure which takes values in such spaces.

A sequence of integrable random variables (f_n) defined on a probability space (S, Σ, p) and adapted to an increasing sequence of σ-fields is called an asymptotic martingale, or, for brevity, an amart if $\lim\limits_{\tau \in T} \int f_\tau dp$ exists, where T is the set of all bounded stopping times directed by the obvious ordering. Martingales as well as submartingales and supermartingales are special cases of amarts.

The concept of an amart was first introduced by Meyer [16] who proved that if a continuous parametered scalar valued amart is essentially bounded then it converges almost everywhere. Using optional sampling techniques, Austin-Edgar-Tulcea [1] proved that if a real-valued amart is L_1-bounded, that is if $\sup\limits_n \int |f_n| dp < \infty$, then (f_n) converges almost everywhere. Chatterji [6] proved that if a martingale is Banach-space-valued and if the Banach space has the Radon-Nikodým property, then it converges almost everywhere if it is L_1-bounded. This is not true for amarts. Chacon and Sucheston [4] proved that if an amart is defined for random variables taking values in a Banach space X which has a separable dual and which has the Radon-Nykodým property and if L_1-boundedness is replaced by the condition $\sup_{\tau \in T} \int |f_\tau| dp < \infty$, then the amart converges almost everywhere in the weak topology of X.

In chapter one we establish basic concepts and terminology concerning amarts of real-valued random variables which are integrable with respect to a probability measure p. In chapter two several convergence theorems are developed for weakly measurable Pettis integrable amarts which take values in a locally convex linear topological vector space. This setting is much more general than that considered by Chacon and Sucheston in [4] and as might be expected additional conditions must be imposed on the amarts to insure convergence. We do not require that the range space be a Banach space nor do we require that it have the Radon-Nykodým property; however we do impose conditions relating to the weak compactness of the image of the amart. We obtain conditions which insure that a discrete-parametered amart converges almost everywhere in the weak topology as well as conditions guaranteeing weak convergence in measure of a continuous-parametered amart. In chapter three similar theorems are obtained for amarts consisting of strongly measurable Pettis integrable random variables. While there are similarities between the conditions imposed on the amarts to insure convergence, the technique of proof is entirely different. Utilizing a technique of Uhl [21] we make crucial use of the isometric isomorphism between the given measure space and the Stone space in which every finitely additive measure is also countably additive. In chapter four a theory of amarts is developed for complex-valued random variables which are integrable with respect to a measure which takes values in a locally convex linear topological space. The integration theory used was developed by Lewis in [13]. Properties of amarts in this setting are studied extensively and an optional sampling theorem as well as several convergence theorems are proved.

CHAPTER I
REAL-VALUED ASYMPTOTIC MARTINGALES

1. Basic Notions

We shall begin by establishing notation and basic concepts for the case of real-valued amarts measurable with respect to a probability measure p. The material found in this chapter constitutes a review of the literature necessary for the understanding of succeeding chapters.

A field $\mathcal{F}$ of subsets of a set S is a nonempty family of subsets of S which is closed under unions and complements. A *σ-field* or *Borel-field* Σ of S is a field which is closed under countable unions. The ordered pair (S, Σ) consisting of a set S and a σ-field Σ of subsets of S is called a *measurable space*. Any function μ from a field $\mathcal{F}$ to the real numbers $\mathbb{R}$, denoted $\mu\colon \Sigma \longrightarrow \mathbb{R}$, is a *real-valued set function*. A set function is *countably additive* if for every disjoint sequence $(E_n) \subseteq \mathcal{F}$, with $\{\cup_n E_n\} \in \mathcal{F}$

$$\mu(\cup_n E_n) = \sum_{n\geq 1} \mu(E_n) \quad ,$$

where the convergence of the infinite series is unconditional. The set function μ is *finitely additive* if the above equality holds for every finite disjoint family $\{E_n\colon n=1,\ldots,N\} \subseteq \mathcal{F}$. A *probability measure* p is a positive countably additive set function defined on a σ-field Σ of a set S such that $p(S) = 1$.

A probability space is a triple (S, Σ, p) where p is a probability measure defined on the σ-field Σ of the set S.

A function $f: S \longrightarrow \mathbb{R}$ is Σ-measurable if for any $c \in \mathbb{R}$ the set $\{s \in S: f(s) < c\}$ belongs to Σ. As is customary in the literature a set of the form $\{s \in S: f(s) < c\}$ will be denoted by $\{f < c\}$. A Σ-measurable function will be called a random variable.

If f is a random variable defined on S then the σ-field generated by f is the smallest σ-field of S which contains all sets of the form $\{f < c\}$, for $c \in \mathbb{R}$. Equivalently it is the smallest σ-field of S containing all sets of the form $\{f \in \mathcal{B}\}$ where $\mathcal{B}$ is a Borel subset of $\mathbb{R}$. Similarly a family $\{f_\alpha: \alpha \in \Lambda\}$ of random variables generates a sub-σ-field of Σ, denoted by $\sigma(\{f_\alpha: \alpha \in \Lambda\})$, which contains all sets of the form $\{f_\alpha \in \mathcal{B}\}$ where $\mathcal{B}$ is a Borel subset of $\mathbb{R}$ and $\alpha \in \Lambda$.

If p is a probability measure defined on a σ-field Σ of S, a p-null set is a set which is contained in a set A of Σ with $p(A) = 0$. The measure p can always be extended by adjoining to Σ all sets A^* such that for some $A \in \Sigma$, $(A-A^*) \cup (A^*-A) \equiv A \Delta A^*$ is a p-null set, and defining $p(A^*) = p(A)$. The measure p thus extended is a complete measure in the sense that if A is a p-null set then it is in Σ and $p(A) = 0$. With p thus extended, a set A is p-null if and only if $p(A) = 0$. We will always assume that a probability measure p is complete.

If the set of points where two random variables f and g are not equal is a p-null set then we say that f = g almost everywhere, abbreviated a.e. .

In the following chapters we will be concerned with three kinds of convergence concepts for sequences of real-valued random variables; almost everywhere convergence, convergence in measure and convergence in mean. The sequence (f_n) converges to f almost everywhere,

$$f_n \rightarrow f \quad \text{a.e.},$$

if there is a p-null set $N \subseteq S$ such that $\lim_n (f_n(s)) = f(s)$ if $s \in S \backslash N$. The sequence (f_n) converges to f in p-measure, or in probability,

$$f_n \rightarrow f \quad \text{in p-measure}$$

if for any $\varepsilon > 0$, $\lim_n (p\{|f_n - f| \geq \varepsilon\}) = 0$. If $(f_n) \subset L_1(p)$ then (f_n) converges to f in mean if the sequence (f_n) converges to f in the L_1-norm; that is,

$$f_n \rightarrow f \quad \text{in } L_1(p)$$

if $\lim_n \int |f_n - f| \, dp = 0$.

2. Conditional Expectation of a Random Variable

Let (S, Σ, p) be a probability space and f a random variable defined on S such that $\int |f| dp < \infty$. For any sub-σ-field Σ' of Σ, the set function $\nu: \Sigma' \rightarrow \mathbb{R}$ defined by

$$\nu(E) = \int_E f dp \quad \text{for } E \in \Sigma'$$

is a countably additive measure such that $\nu(E) \to 0$ as $p(E) \to 0$. Any measure satisfying the above condition is said to be absolutely continuous with respect to p, in symbols $\nu << p$. By the Radon-Nykodým theorem there is a Σ'-measurable, integrable function g such that for $E \in \Sigma'$,

$$\nu(E) = \int_E g dp = \int_E f dp .$$

Any Σ'-measurable function which is p-a.e. equal to the above function g is called the conditional expectation of f given Σ' and is denoted by $E[f|\Sigma']$. Let $E \in \Sigma$ and denote by I_E the random variable that is one if $s \in E$ and zero otherwise. The function I_E will be called the indicator function of E. The conditional expectation of I_E given Σ' is called the conditional probability of E given Σ' and is denoted by $p(E|\Sigma')$.

Let Σ' be the sub-σ-field of Σ generated by a family $\{f_\alpha: \alpha \in \Lambda\}$ of random variables. Then $E[f|\Sigma']$, denoted by $E[f|f_\alpha: \alpha \in \Lambda\}$, is called the conditional expectation of f given $\{f_\alpha: \alpha \in \Lambda\}$. Intuitively $E[f|f_\alpha: \alpha \in \Lambda]$ is the "best estimate" of the random variable f given the information contained in knowledge of the random variables $\{f_\alpha: \alpha \in \Lambda\}$. To elaborate, suppose $\Lambda = \{1,\ldots,n\}$ and $f,f_1,\ldots,f_n$ are square integrable random variables. Then $E[f|f_1,\ldots,f_n]$ is the estimate of f which minimizes $\int|\Psi(f_1(s),\ldots,f_n(s)) - f(s)|^2 dp$ where Ψ ranges over all Borel-measurable functions defined on $\mathbb{R}^N$ such that $\int(\Psi(f_1(s),\ldots,f_n(s)) - f(s))dp = 0$. Considering the functions $\{f,f_1,\ldots,f_n\}$ as elements of the Hilbert space $L_2(S, \Sigma, p)$, $E[f|f_1,\ldots,f_n]$ is the orthogonal projection of f on $L_2(S, \sigma(f_1,\ldots,f_n), p)$. See [14] for further details.

3. Martingales

Let (f_n) be a sequence of random variables and (Σ_n) an increasing sequence of sub-σ-fields of Σ. For example, Σ_n could be the σ-field generated by $\{f_1,\ldots,f_n\}$ for $n \in Z$, where Z denotes the positive integers. The sequence (f_n) is said to be adapted to (Σ_n) if f_n is Σ_n-measurable for each $n \in Z$. A martingale, $(f_n, \Sigma_n, n \in Z)$ is a sequence (f_n) of integrable random variables adapted to an increasing sequence (Σ_n) of sub-σ-fields of Σ such that

$$E[f_n|\Sigma_m] = f_m$$

for every $n, m \in Z$, $m \le n$. It is a submartingale if

$$E[f_n|\Sigma_m] \ge f_m$$

and a supermartingale if

$$E[f_n|\Sigma_m] \le f_m$$

for every $m \le n$. Note that $(f_n, \Sigma_n, n \in Z)$ is a supermartingale if and only if $(-f_n, \Sigma_n, n \in Z)$ is a submartingale.

A martingale is said to be L_1-bounded if it is bounded in the L_1-norm; that is if

$$\sup \int |f_n| dp < \infty.$$

Doob's classical martingale convergence theorem states that if $(f_n, \Sigma_n, n \in Z)$ is an L_1-bounded martingale then $\lim_n f_n = f$ p-a.e. for some $f \in L_1(p)$. This convergence theorem and related martingale techniques have found widespread application. For instance, particularly elegant proofs of the Zero-one law and the Strong Law of Large Numbers can be derived using martingale convergence theorems.

If f_∞ is an integrable random variable and (Σ_n) is an increasing sequence of sub-σ-fields of Σ, a martingale $(f_n, \Sigma_n, n \in Z)$ can be formed by defining $f_n = E[f_\infty | \Sigma_n]$ for each $n \in Z$. The fact that this is a martingale follows from the fact that

$$E[E[f_\infty | \Sigma_n] | \Sigma_m] = E[f_\infty | \Sigma_m]$$

if $\Sigma_m \subseteq \Sigma_n$. Any martingale which can be manufactured in the above fashion is called a complete martingale. Note that if $(f_n, \Sigma_n, n \in Z)$ is a complete martingale then $(f_n, \Sigma_n, n=1,\ldots,\infty)$ is also a martingale. It is not hard to prove that a martingale is complete if and only if it is uniformly integrable; that is, if and only if

$$\lim_{c \to \infty} \int_{\{|f_n| > c\}} |f_n| dp = 0$$

uniformly in n.

Complete martingales are particularly well behaved as far as convergence is concerned. In [8, Theorem 4.1] Doob shows that a complete martingale converges in L_1-norm, as well as p-a.e., to f_∞.

In 1973 C. W. Lamb [12] gave a particularly short proof of Doob's martingale convergence theorem for L_1-bounded martingales by showing that any L-bounded martingale can be approximated by a uniformly integrable martingale and then using Doob's maximal inequality

$$p\{\sup_n |f_n| \geq \lambda\} \leq \frac{1}{\lambda} \sup_n \int |f_n| dp, \ \lambda > 0,$$

to show that a uniformly integrable martingale converges p-a.e.. This technique coupled with optional sampling techniques will prove to be very useful in the following chapters.

4. Asymptotic Martingales

We continue to assume that (f_n) is a sequence of random variables adapted to an increasing sequence (Σ_n) of sub-σ-fields and that $\sigma(\bigcup_{n\geq 1} \Sigma_n) = \Sigma$. A stopping time is a random variable $\tau: S \rightarrow \{0, 1, \ldots, \infty\}$ such that for each $n \in Z$, $\{\tau=n\} \in \Sigma_n$. The set of all bounded stopping times is denoted by T. If $\tau \in T$ then a random variable f_τ is defined by

$$f_\tau(s) = f_{\tau(s)}(s);$$

in other words $f_\tau(s) = f_n(s)$ on the set $\{\tau=n\}$, $n \in Z$.

We can define a partial ordering on T by defining $\tau_1 \leq \tau_2$ for $\tau_1, \tau_2 \in T$ if $\tau_1(s) \leq \tau_2(s)$ for p-a.e. $s \in S$. With this partial ordering $(T, \leq)$ becomes a directed set. If $(f_n, \Sigma_n, n \in Z)$ is a martingale and $\tau \in T$, then there is an integer K such that $\tau \leq K$. Consequently

$$\int f_\tau dp = \sum_{\lambda\le K} \int_{\{\tau=\lambda\}} f_\lambda dp = \sum_{\lambda\le K} \int_{\{\tau=\lambda\}} f_K dp = \int f_K dp = \int f_1 dp.$$

Hence $\int f_\tau dp$ does not depend on the choice of $\tau \in T$. To quote Chacon and Sucheston in [4], "it seems therefore natural to call $(f_n, \Sigma_n, n \in Z)$ an asymptotic martingale if $\lim_T \int f_\tau dp$ exists." If (f_n) is a sequence of real-valued random variables defined on a probability space (S, Σ, p), $(f_n, \Sigma_n, n \in Z)$ is an asymptotic martingale or amart if (f_n) is adapted to the increasing sequence of sub-σ-fields (Σ_n) of Σ and

$$\lim_{\tau\in T} \int f_\tau dp$$

exists. In the following chapters amarts of vector-valued random variables integrable with respect to a probability measure and scalar-valued amarts integrable with respect to a vector-valued measure will be defined analogously.

Amarts generalize martingales considerably since every convergent sequence of random variables with integrable supremum is an amart. If $(f_n, \Sigma_n, n \in Z)$ is a submartingale and $(\int f_n dp\colon n \in Z)$ is bounded then $(f_n, \Sigma_n, n \in Z)$ is an amart since for $\tau \le \sigma \in T$ there is an integer K such that $\tau \le K$. Consequently

$$\int f_\tau dp = \sum_{i\le K} \int_{\{\tau=i\}} f_i dp \le \sum_{i\le K} \int_{\{\tau=i\}} f_\sigma dp = \int f_\sigma dp .$$

(The inequality follows from the submartingale property and the fact that $\sigma \ge i$ on the set $\{\tau=\lambda\}$.) Similarly if $(f_n, \Sigma_n, n \in Z)$ is a supermartingale and $(\int f_n dp\colon n \in Z)$ is bounded then $(f_n, \Sigma_n, n \in Z)$ is an amart.

As noted in [2], amarts combine several useful properties of martingales, submartingales and supermartingales. For example, while martingales are closed under linear combinations, supermartingales are closed under infimum and submartingales are closed under supremum, the class of amarts is closed under all three operations. As pointed out by Chacon and Sucheston in [4] amarts have a possible pedagogical interest. This stems from the fact that the Radon-Nikodým theorem can be derived from the amart convergence theorem just as it is derived from the martingale convergence theorem in Meyer [16, P. 153]. However, to derive the Radon-Nikodým theorem by the martingale convergence theorem one must define conditional expectation without the Radon-Nikodým theorem. The amart treatment, on the other hand, does not involve the conditional expectation which can then be defined after the Radon-Nikodým theorem is proved.

CHAPTER II
ASYMPTOTIC MARTINGALES OF WEAKLY MEASURABLE, PETTIS INTEGRABLE RANDOM VARIABLES

In this chapter we define amarts consisting of weakly measurable functions defined on a probability space (S, Σ, p) with values in a locally convex linear topological space V over the complex numbers. We obtain several theorems which specify conditions which are sufficient to insure convergence of these amarts in the weak topology of V.

A linear topological space V is said to be locally convex if it possesses a base for its topology consisting of convex sets. The continuous dual of V will be denoted by V^*. If $x \in V$ and $x^* \in V^*$, then the action of x^* on x is denoted by $\langle x^*,x\rangle$ or where the meaning is clear simply by x^*x. A function f from S to V is a weakly measurable random variable if for each $x^* \in V^*$ the complex valued function

$$x^*f(s) = x^*(f(s))$$

is a measurable function. The integral used in this chapter is that of Pettis [17]. A random variable $f: S \to V$ is Pettis integrable, sometimes called weakly integrable, if

$$\int |x^*f|\,dp < \infty$$

for every $x^* \in V^*$, and for each $E \in \Sigma$ there is an element of V, denoted by $\int_E f dp$, such that

$$x^*(\int_E f dp) = \int_E x^* f dp$$

for every $x^* \in V^*$. As in the previous chapter $(\Sigma_n : n \in Z)$ will denote an increasing sequence of sub-σ-fields of Σ and T will denote the set of all bounded stopping times. If a sequence (f_n) of weakly measurable, Pettis integrable random variables is adapted to $(\Sigma_n : n \in Z)$ then $(f_n, \Sigma_n : n \in Z)$ will be called an amart if

$$\lim_{\tau \in T} \int f_\tau dp$$

exists with respect to the topology on V.

The map $x \to \hat{x}$, where $\hat{x}$ is defined by $\hat{x}(x^*) = x^* x$, is an embedding of V into the space of linear functionals on V^*. Thus V can be considered as a linear subspace of the space of all linear functionals on V^*. It is important to know what conditions, in particular what types of topologies on V^*, insure that V is mapped onto the continuous dual $(V^*)^*$ of V^*. The Mackey-Arens theorem provides such conditions. Before stating this theorem a few definitions concerning a linear topological space V will be needed.

Definition 2.1. A set A in V is balanced if $\lambda A \subset A$ for every complex number λ such that $|\lambda| \leq 1$.

Definition 2.2. Let $\underline{G}$ be a collection of subsets of V; then the $\underline{G}$-topology of V^* is the topology of uniform convergence on sets of $\underline{G}$.

Thus a net $(x_\alpha^*: \alpha \in \Lambda) \subset V^*$ converges to $x^* \in V^*$ in this topology if and only if for $\epsilon > 0$ and $Q \in \underline{G}$ there is a $\alpha_0 \in \Lambda$ such that if $\alpha \geq \alpha_0$ then

$$|x_\alpha^*(x) - x^*(x)| < \epsilon$$

for all $x \in Q$.

Definition 2.3. The weak topology of V is the topology obtained by taking as a base all sets of the form

$$\{y \in V: |x^*(x) - x^*(y)| < \epsilon, x^* \in A\},$$

where $x \in V$, A is a finite collection of continuous linear functionals in V^* and $\epsilon > 0$.

Therefore a net $(x_\alpha: \alpha \in \Lambda) \subset V$ converges to $x \in V$ in the weak topology if and only if

$$\lim_\alpha x^*(x_\alpha) = x^*(x)$$

for every $x^* \in V^*$.

A subset of V which is compact with respect to this topology is said to be weakly compact.

Theorem 2.4. (Mackey-Arens). [11, Theorem 3.5.1] Let Γ be a locally convex topology on V^*; then the continuous dual of V^* with respect to the topology Γ can be identified with V, i.e. $(V^*)^* = V$, if and only if Γ is a $\underline{G}$-topology where $\underline{G}$ is some collection of balanced, convex, weakly compact subsets of V which cover V.

We shall need to use the following theorem for real-valued amarts which was proved by Austin, Edgar and Tulcea in [1].

Theorem 2.5. [1, p.18] If $(f_n, \Sigma_n, n \in Z)$ is a real-valued amart such that

$$\sup_{n \in Z} \int |f_n| dp < \infty ,$$

then the sequence (f_n) converges almost everywhere to an integrable random variable f.

If in the above theorem the boundedness condition is replaced by the stronger condition of uniform integrability, that is if for any $\epsilon > 0$ there is a real number b such that

$$\int_{\{f_n > b\}} |f_n| dp < \epsilon ,$$

for all $n \in Z$, then the Vitali Convergence Theorem [9] implies that in addition to converging almost everywhere the sequence (f_n) converges in the L_1-norm to an integrable random variable f. We state this as a corollary to the above theorem.

Corollary 2.6. If the amart $(f_n, \Sigma_n: n \in Z)$ is uniformly integrable then the sequence (f_n) converges to an integrable random variable f in the L_1-norm as well as almost everywhere.

The following definitions and basic facts concerning filters will be needed in the proof of the next theorem. A filter $\mathcal{F}$ on a set S is a nonempty collection of nonempty sets of S such that

(i) if F_1, $F_2 \in \mathcal{F}$ then $F_1 \cap F_2 \in \mathcal{F}$,

(ii) if $F \in \mathcal{F}$ and $F \subset F'$, then $F' \in \mathcal{F}$.

A subcollection $\mathcal{B}$ of $\mathcal{F}$ is a filter base for $\mathcal{F}$ if and only if each member of $\mathcal{F}$ contains some member of $\mathcal{B}$. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are two filters on S, $\mathcal{F}_1$ is said to be finer than $\mathcal{F}_2$ if $\mathcal{F}_1 \supset \mathcal{F}_2$. A filter $\mathcal{F}$ is called an ultrafilter if there is no strictly finer filter than $\mathcal{F}$. An easy application of Zorn's Lemma establishes that every filter is contained in some ultrafilter.

If V is a topological space and $x \in V$, the collection $\mathcal{B}(x)$ of all neighborhoods of x forms a filter on V called the neighborhood filter at x. A filter $\mathcal{F}$ on V is said to converge to x if $\mathcal{F}$ is finer than $\mathcal{B}(x)$. If a filter $\mathcal{F}_1$ is finer than $\mathcal{F}_2$ and the filter $\mathcal{F}_2$ converges to x then it is clear that $\mathcal{F}_1$ also converges to x. If the filter $\mathcal{F}_1$ converges to a point x then x is said to be the limit of $\mathcal{F}_1$. Denote this by $\lim (\mathcal{F}_1) = x$. A filter base $\mathcal{B}$ converges to x if the filter generated by $\mathcal{B}$ converges to x. If f is a map from a set X to a set Y and if $\mathcal{F}$ is a filter on X, then f maps the filter $\mathcal{F}$ onto a filter base $\{f(F): F \in \mathcal{F}\}$.

Denote the filter generated by this base by $f(\mathcal{J})$. If X and Y are topological spaces, then f is continuous at $x \in X$ if and only if whenever $\lim (\mathcal{J}) = x$ then $\lim [f(\mathcal{J})] = f(x)$.

Associated with any sequence (or net) (x_n) in V is a filter formed by taking as a base the sets $S_M = \{x_n : n \geq M\}$ for $M \in Z$. This filter is called the elementary filter, or Frechét filter, associated with (f_n). The sequence (x_n) converges to the point x if and only if the elementary filter associated with (x_n) converges to x. This is clear since both conditions mean that each neighborhood of x contains a set S_M.

The following characterization of compactness will be used in the next theorem. Its proof can be found in any book on elementary topology.

Lemma 2.7. A topological space V is compact if and only if every ultrafilter on V converges.

In [15] Métivier obtained conditions which insure almost everywhere convergence of a weakly integrable weakly measurable martingale. Using related techniques we obtain similar conditions insuring convergence of an amart.

Theorem 2.8. Let $(f_n, \Sigma_n : n \in Z)$ be a weakly measurable, weakly integrable amart such that the following conditions are true.

(a) For each $x^* \in V^*$, $\sup_{n \in Z} \int |x^* f_n| dp < \infty$.

(b) For almost all $s \in S$, there exists a balanced, convex, weakly compact subset Q_s of V such that $\cup_{n\geq 1}\{f_n(s)\} \subset Q_s$.

(c) The space V^* is separable with respect to a $\underline{G}$-topology with $Q_s \in \underline{G}$ for $s \in S$.

Then there is a weakly measurable, Pettis integrable random variable f_∞ such that for almost all $s \in S$,

$$\lim_n x^*f_n(s) = x^*f_\infty(s)$$

for every $x^* \in V^*$.

Proof. For each $x^* \in V^*$, $(x^*f_n, \Sigma_n: n \in Z)$ is an amart such that $\sup_{n\ Z} \int|x^*f_n|dp < \infty$. Therefore by theorem 2.5 the sequence (x^*f_n) converges almost everywhere to an integrable random variable h_{x^*}. The proof will therefore be complete if we establish the existence of a random variable $f_\infty: S \rightarrow V$ such that for all $x^* \in V^*$, and almost all $s \in S$, $x^*f_\infty(s) = h_{x^*}(s)$.

By condition (b), for almost all $s \in S$ there is a balanced, convex, weakly compact subset Q_s of V such that $\cup_{n\geq 1}\{f_n(s)\} \subset Q_s$. Let $\mathcal{J}_s$ be an ultrafilter which is finer than the elementary filter associated with the sequence $(f_n(s))$. Since the ultrafilter $\mathcal{J}_s$ is contained in a weakly compact set Q_s, Lemma 2.7 implies that $\mathcal{J}_s$ converges in the weak topology to some element of V. Define $f_\infty(s)$ to be the limit of $\mathcal{J}_s$ for those s for which the above limit exists and zero otherwise. Then for all $x^* \in V^*$ and almost all $s \in S$

$$\lim [x^*(\mathcal{J}_s)] = x^*f_\infty(s) .$$

Therefore since for each $x^* \in V^*$ we have

$$\lim_n (x^*f_n(s)) = h_{x^*}(s) \quad \text{a.e.} ,$$

and $x^*(\mathcal{J}_s)$ is a finer filter than the elementary filter associated with the sequence $(x^*f_n(s))$, we have

$$h_{x^*}(s) = \lim_n x^*f_n(s) = \lim [x^*(\mathcal{J}_s)] = x^*f_\infty(s) \quad \text{a.e.} .$$

Let $A = \{x_k^*: k \in Z\}$ be a dense subset of V^* in the $\underline{G}$-topology of V^*. Then for each $x_k^* \in A$ there is a P-null set F_k such that if $s \notin F_k$ then

$$h_{x_k^*}(s) = \lim_n (x_k^*f_n(s)) = x_k^*f_\infty(s) .$$

Thus if s is not in the null set $F = \bigcup_{k\in Z} F_k$, then

$$h_{x_k^*}(s) = x_k^*f_\infty(s)$$

for $x_k^* \in A$.

Since A is dense in the $\underline{G}$-topology of V^*, for any $\epsilon > 0$ and $x^* \in V^*$ if $s \notin F$, there is an $x_k^* \in A$ such that x_k^* is contained in the neighborhood

$$\{y^*: |y^*(t) - x^*(t)| < {}^{\epsilon}/3 \text{ for all } t \in Q_s\}$$

of x^*. Choose an integer N such that if $n \geq N$ then

$$|x_k^* f_n(s) - x_k^* f_\infty(s)| < \epsilon/3 .$$

Then for $n \geq N$

$$\begin{aligned}|x^* f_n(s) - x^* f_\infty(s)| &\leq |x^* f_n(s) - x_k^* f_n(s)| \\ &+ |x_k^* f_n(s) - x_k^* f_\infty(s)| \\ &+ |x_k^* f_\infty(s) - x^* f_\infty(s)| \\ &< \epsilon .\end{aligned}$$

Thus for all $x^* \in V^*$ and almost all $s \in S$ we have

$$h_{x^*}(s) = \lim_n x^* f_n(s) = x^* f_\infty(s) ,$$

and the proof is complete.

Q.E.D.

Just as for a sequence of random variables, if $(f_\alpha\colon \alpha \in \Lambda)$ is a net of weakly measurable, weakly integrable random variables adapted to an increasing net $(\Sigma_\alpha\colon \alpha \in \Lambda)$ of sub-σ-algebras of Σ, then $(f_\alpha, \Sigma_\alpha\colon \alpha \in \Lambda)$ is an amart if the net $(\int f_\tau dp\colon \tau \in T)$ converges in the topology of V. The set of bounded stopping times T now refers to the set of all functions $\tau\colon S \to \Lambda$ with τ taking only finitely many values and such that $\{\tau = \alpha\} \in \Sigma_\alpha$ for all $\alpha \in \Lambda$.

For real valued amarts it is not difficult to extend Theorem 2.5 to the case where the amart is a net rather than a sequence if almost everywhere convergence is replaced by convergence in probability. This was done by Edgar and Sucheston [10].

Theorem 2.9. [10, p.206] Let Λ be a directed set and $(f_\alpha, \Sigma_\alpha: \alpha \in \Lambda)$ a real-valued amart such that $\sup_{\alpha\in\Lambda} \int |f_\alpha| dp < \infty$. Then the net $(f_\alpha: \alpha \in \Lambda)$ converges in probability.

Theorem 2.10. Let V be a locally convex linear topological space with dual V*. Let G be a family of convex, balanced weakly compact subsets of V. Suppose the net of weakly measurable, weakly integrable random variables $(f_\alpha: \alpha \in \Lambda)$ defined on S and taking values in V is an amart adapted to an increasing net $(\Sigma_\alpha: \alpha \in \Lambda)$ of sub-σ-algebras of Σ. Then there exists a weakly measurable, weakly integrable random variable f_∞ such that $\lim_\alpha f_\alpha = f_\infty$ weakly in measure if either of the following conditions hold.

(a) For each $x^* \in V^*$

$$\sup_{\alpha\in\Lambda} \int |x^* f_\alpha| dp < \infty$$

and for $\varepsilon > 0$ there exists $\Omega_\varepsilon \in \Sigma_\infty$ and $Q_\varepsilon \in \underline{G}$ such that $p(S\setminus\Omega_\varepsilon) < \epsilon$ and $I_{\Omega_\varepsilon} \circ f_\alpha(s) \in Q_\varepsilon$ a.e. for all $\alpha \in \Lambda$.

(b) There exists a real valued amart $(h_\alpha, \Sigma_\alpha: \alpha \in \Lambda)$ such that

$$\sup_{\alpha\in\Lambda} \int |h_\alpha| dp < \infty$$

and a $Q \in \underline{G}$ such that $f_\alpha(s) \in h_\alpha(s)Q$ a.e. for all $\alpha \in \Lambda$.

Proof. Note that in both cases we have

$$\sup_{\alpha\in\Lambda} \int |x^*f_\alpha| dp < \infty .$$

In (a) this is by hypothesis. In (b) it follows from the observation that the weak compactness of Q implies that $\sup_{q\in Q} |x^*q| < \infty$, and thus since $f_\alpha(s) \in h_\alpha(s) \cdot Q$ a.e., we have

$$|x^*f_\alpha(s)| \le \sup_{q\in Q} |x^*q| \cdot |h_\alpha(s)| \quad \text{a.e.} \quad .$$

Therefore

$$\sup_{\alpha\in\Lambda} \int |x^*f_\alpha| dp \le (\sup_{q\in Q} |x^*q|) \cdot (\sup_{\alpha\in\Lambda} \int |h_\alpha| dp) < \infty .$$

Since $(x^*f_\alpha)_{\alpha\in\Lambda}$ is a real valued amart for each $x^* \in V^*$, it follows from Theorem 2.9 that $(x^*f_\alpha : \alpha \in \Lambda)$ converges in measure to some real valued integrable function which we shall denote by g_{x^*}. The proof will therefore be complete after we establish the existence of a random variable $f_\infty : S \to V$ such that for all $x^* \in V^*$ and a.e. $s \in S$, $x^*f_\infty(s) = g_{x^*}(s)$.

Assume condition (a) is satisfied. It is then possible to choose an increasing sequence (Ω'_n) of sets in Σ such that $p(\bigcup_{n\ge 1} \Omega'_n) = p(S)$ and a corresponding sequence (Q_n) of sets in $\underline{G}$ such that $I_{\Omega'_n} \circ f_\alpha(s) \in Q_n$ a.e. for all $\alpha \in A$. Define

$$\Omega_1 = \Omega'_1,\ \Omega_2 = \Omega'_2\backslash\Omega_1, \ldots,\ \Omega_n = \Omega'_n\backslash\Omega_{n-1}, \ldots$$

so that the sets $(\Omega_n : n \in Z)$ are disjoint.

Since

$$x^*(I_{\Omega_n} \circ f_\alpha(s)) \le \sup_{q\in Q_n} |x^*q| < \infty$$

independently of α a.e., and since $x^*f_\alpha \xrightarrow{\alpha} g_{x^*}$ in measure, it follows that

$$|I_{\Omega_n} \circ g_{x^*}(s)| \le \sup_{q\in Q_n} |x^*q| < \infty \text{ a.e.} \quad .$$

Therefore

$$I_{\Omega_n} \circ g_{x^*} \in L^\infty(S, \Sigma, p) \quad .$$

The map $x^* \rightarrow I_{\Omega_n} \circ g_{x^*}$ is a linear map from V^* into $L^\infty(S, \Sigma, p)$ which is continuous with respect to the $\underline{G}$-topology of V^*. To see that it is continuous note that if $\sup_{q\in Q_n} |x^*q| \le \delta$ for some $\delta > 0$, then $|x^*(I_{\Omega_n} \circ f_\alpha(s))| \le \delta$ a.e. for all $\alpha \in \Lambda$. This implies that $|I_{\Omega_n} \circ g_{x^*}(s)| \le \delta$ a.e., and thus $\|I_{\Omega_n} \circ g_{x^*}\|_\infty \le \delta$. Let $(x^*_\ell : \ell \in L)$ be a net which converges to x^* in the $\underline{G}$-topology of V^*. Then for $\varepsilon > 0$ and $Q_n \in \underline{G}$ there exists a $\gamma \in L$ such that if $\ell \ge \gamma$ then $|x^*_\ell(q) - x^*(q)| < \varepsilon$ for all $q \in Q_n$. Therefore

$$\langle x^*_\ell - x^*, I_{\Omega_n} \circ f_\alpha(s)\rangle < \varepsilon$$

for almost all $s \in S$. Consequently

$$\|I_{\Omega_n} \circ g_{x^*_\ell - x^*}\|_\infty < \varepsilon,$$

which by linearity implies that

$$\|I_{\Omega_n} \circ g_{x^*_\ell} - I_{\Omega_n} \circ g_{x^*}\|_\infty < \varepsilon \quad .$$

We have thus demonstrated that if $\lim_{\ell \in L} (x^*_\ell) = x^*$ in the $\underline{G}$-topology of V^* then $\lim_{\ell \in L} (I_{\Omega_n} \circ g_{x^*_\ell}) = I_{\Omega_n} \circ g_{x^*}$ in $L^\infty(S, \Sigma, p)$.

Let $B(S,\Sigma)$ denote the space of Σ-measurable, bounded functions h with norm $\|h\| = \sup_{s \in S} |h(s)|$. By the lifting theorem of A. and C. Ionescu Tulcea [20, Theorem IV,3] there exists a continuous linear function ρ,

$$\rho\colon L^\infty(S, \Sigma, p) \to B(S, \Sigma) \ ,$$

such that for every $h \in L^\infty(S, \Sigma, p)$ we have $\rho(h) \in h$ and $\|\rho(h)\| \le \|h\|_\infty$. (Recall that an element of $L^\infty(S, \Sigma, p)$ is actually an equivalence class of functions.)

Denote $\rho(I_{\Omega_n} \circ g_{x^*})$ by $g^{\Omega_n}_{x^*}$. Then the map

$$x^* \to g^{\Omega_n}_{x^*}(s)$$

is a continuous linear functional with respect to the $\underline{G}$-topology on V^*. By Theorem 2.4 when V^* is endowed with the $\underline{G}$-topology, V is equivalent to the continuous dual of V^*. Therefore since the map $x^* \to g^{\Omega_n}_{x^*}(s)$ is continuous with respect to the $\underline{G}$-topology on V^*, it is an element of V. Denote it by $f^{\Omega_n}_\infty(s)$. Then $f^{\Omega_n}_\infty\colon S \to V$, $f^{\Omega_n}_\infty(s) = 0$ if $s \notin \Omega_n$ and $x^* f^{\Omega_n}_\infty(s) = g^{\Omega_n}_{x^*}(s) = \rho(I_{\Omega_n} \circ g_{x^*})(s) = I_{\Omega_n} \circ g_{x^*}(s)$

for all $x^* \in V^*$ and a.e. $s \in S$. Consequently if f_∞ is defined by

$$f_\infty(s) = f_\infty^{\Omega_n}(s) \text{ if } s \in \Omega_n$$
$$= 0 \text{ if } s \notin \bigcup_{n\geq 1} \Omega_n$$

then

$$x^*f_\infty(s) = g_{x^*}(s) \text{ for all } x^* \in V^* \text{ and a.e. } s \in S$$

and the proof is complete for part (a).

If condition (b) is true, an application of Theorem 2.9 shows that the net $(h_\alpha : \alpha \in \Lambda)$ converges in probability to an integrable random variable h_∞. By the integrability of h_∞ it is possible to construct a disjoint sequence of sets $(\Omega_n : n \geq 1)$ such that $p(\bigcup_{n\geq 1} \Omega_n) = 1$ and $|h_\infty(s)| < n$ if $s \in \Omega_n$. As in the proof of part (a), for each $x^* \in V^*$

$$x^*I_{\Omega_n} \circ f_\alpha \xrightarrow{\alpha} I_{\Omega_n} \circ g_{x^*} \text{ in measure.}$$

By hypothesis there is a weakly compact subset $Q \in \underline{G}$ such that $f_\alpha(s) \in h_\alpha(s)Q$ for almost all $s \in S$. Consequently, for each $x^* \in V^*$ and $\alpha \in \Lambda$,

$$|x^*I_{\Omega_n} \circ f_\alpha(s)| \leq I_{\Omega_n}(s) \cdot |h_\alpha(s)| \cdot (\sup_{q\in Q} |x^*q|) .$$

Since $I_{\Omega_n} \circ h_\alpha \xrightarrow{\alpha} h_\infty$ in measure and $|h_\infty(s)| < n$ for $s \in \Omega_n$ it follows that

$$|(I_{\Omega_n} \circ g_*)(s)| \leq I_{\Omega_n}(s) \cdot |h_\infty(s) \cdot (\sup_{q\in Q} |x^*q|)$$
$$\leq n \cdot (\sup_{q\in Q} |x^*q|) \quad \text{a.e.}$$

Therefore

$$\| I_\Omega \circ g_x \|_\infty \le n \cdot (\sup_{q \in Q} |x^*q|) < \infty \quad .$$

The proof now follows as in part (a).

Q.E.D.

CHAPTER III
ASYMPTOTIC MARTINGALES OF STRONGLY MEASURABLE, PETTIS INTEGRABLE RANDOM VARIABLES

In this chapter convergence theorems are investigated for amarts consisting of both sequences and nets of strongly measurable, Pettis integrable random variables which take values in a locally convex linear topological space V. Results are obtained which extend the work of Chacon and Sucheston who in [4] investigated convergence properties of amarts consisting of sequences of Banach-valued Bochner integrable random variables.

Several preliminary theorems and definitions will be needed. Let S be an arbitrary set and $\mathcal{T}$ a field of subsets of S. The space $B(S,\mathcal{T})$ consists of all uniform limits of finite linear combinations of indicator functions of sets in Σ. The space B(S) consists of all bounded scalar functions on S. If S is a topological space C(S) denotes the space of all bounded continuous scalar functions on S. All of the above spaces are Banach spaces when endowed with the norm

$$\| f \| = \sup \{|f(s)| : s \in S\} \quad .$$

Furthermore the above spaces are all algebras, for if f and g are contained in any of the above spaces then the product $f \cdot g$ defined by

$$(f \cdot g)(s) = f(s) \cdot g(s) \quad , \; s \in S$$

is also in the space.

For the proofs of the next three theorems the reader is referred to chapter IV of Dunford and Schwartz [9].

Theorem 3.1. [9, p. 276] Let the closed subalgebra $\mathfrak{U}$ of the real algebra B(S) contain the unit. Then there is a compact Hausdorff space S_1 and an isometric isomorphism between the real algebra $\mathfrak{U}$ and $C(S_1)$.

Theorem 3.2. [9, p. 312] Let S_1 be a compact Hausdorff space such that $B(S,\mathcal{J})$ is isometrically isomorphic with $C(S_1)$. Then the space S_1 is uniquely determined to within a homeomorphism and is totally disconnected. The correspondence $I_E \rightarrow I_{E_1}$ establishes an isomorphism γ of the field $\mathcal{J}$ onto the field $\mathcal{J}_1$ of all open and closed sets in S_1.

Theorem 3.3. [9, p. 312] Let $B(S,\mathcal{J})$ be isometrically isomorphic with $C(S_1)$.

(a) There is an isometric isomorphism T of $ba(S,\mathcal{J})$ onto $ba(S_1,\mathcal{J}_1)$ determined by the correspondence $(T\mu)(E_1) = \mu(\gamma^{-1}(E_1))$ for μ in $ba(S,\mathcal{J})$ and E_1 in $\mathcal{J}_1$.

(b) Each $\mu_1 \in ba(S_1, \mathcal{J}_1)$ has a unique extension to a regular countably additive measure μ_2 in $ca(S_1, \mathcal{J}_2)$ where $\mathcal{J}_2$ is the σ-field generated by $\mathcal{J}_1$. Each μ_2 in $ca(S_1, \mathcal{J}_2)$ is regular. The correspondence $U: \mu_1 \rightarrow \mu_2$ is an isometric isomorphism of $ba(S_1, \mathcal{J}_1)$ onto $ca(S_1, \mathcal{J}_2)$.

(c) If E_1 is in $\mathcal{J}_1$ then the total variation of μ_1 on E_1 is equal to the total variation of $U(\mu_1)$ on E_1 for all μ_1 in $ba(S_1, \mathcal{J}_1)$.

If f is a strongly measurable, Pettis integrable function from (S, Σ, p) to a Banach space X, a norm can be defined as in [17] by

$$\|f\|_{P_1} = \sup_{\|x^*\| \leq 1} \int |x^*f| dp \quad .$$

The space of strongly measurable Pettis integrable functions with this norm will be denoted by $P_1(p,X)$. This space is not complete. (See [17].)

The next three theorems from Pettis [17] will be needed in the development of convergence theorems for amarts consisting of functions from $P_1(p,X)$.

Theorem 3.4. [17, p. 291] A function f is in $P_1(p,X)$ if and only if there exist simple functions $\{f_n: n \in Z\} \subset P_1(p,X)$ such that $f_n(s) \xrightarrow{n} f(s)$ μ-a.e. and $\| f_n - f \|_{P_1} \xrightarrow{n} 0$.

Lemma 3.5. [17, p. 286] A sequence $(f_n) \subset P_1(p,X)$ is a Cauchy sequence if and only if

$$\sup \{\| \int_E (f_n - f_m) dp \|_X : E \in \Sigma\} \longrightarrow 0$$

as n, m $\longrightarrow \infty$.

Theorem 3.6. [17, p. 290] If a sequence $(f_n) \subset P_1(p,X)$ converges weakly in measure to f, and if $\lim_n (\int_E f_n dp)$ exists for every set E in Σ, then f is in $P_1(p,X)$ and

$$\lim_n (\int_E f_n dp) = \int_E f dp$$

for every set E in Σ.

For a directed set Λ, let $\{f_\alpha: \alpha \in \Lambda\}$ be a net of functions from $P_1(p,X)$ and $\{\Sigma_\alpha: \alpha \in \Lambda\}$ an increasing net of sub-σ-fields of Σ. Then as in the previous chapters $\{f_\alpha, \Sigma_\alpha,: \alpha \in \Lambda\}$ will be called an amart if $\lim_{\tau \in T} \int f_\tau dp$ exists in X. The set of bounded stopping times T is defined to be the set of all functions $\tau: S \rightarrow \Lambda$ with τ taking only finitely many values and such that $\{\tau = \alpha\} \in \Sigma_\alpha$ for all $\alpha \in \Lambda$. If the field B is defined by

$$B \equiv \cup \{\Sigma_\alpha: \alpha \in \Lambda\}$$

then we may assume that $\sigma(B) = \Sigma$. In other words Σ is equal to the σ-field generated by B. By applying Theorem 3.1 and Theorem 3.2 we obtain a totally disconnected compact Hausdorff space S_1 such that B_1, the field of all open, closed subsets of S_1, is isomorphic to B under a Boolean isomorphism $\gamma: B \rightarrow B_1$, i.e., $\gamma(E \cup F) = \gamma(E) \cup \gamma(F)$, $\gamma(E \cap F) = \gamma(E) \cap \gamma(F)$, and $\gamma(E^c) = (\gamma(E))^c$ for all E, F in B. As a consequence of Theorem 3.3 every finitely additive real valued set function on B_1 is countably additive. If p_1 is defined on B_1 by

$$p_1(\gamma(E)) = p(E)$$

for all E in B, then since p_1 is countably additive it can be uniquely extended to $\sigma(B_1)$. The σ-field $\sigma(B_1)$ will be denoted by Σ_1.

Since both (S, Σ, p) and (S_1, Σ_1, p_1) are measure spaces (p and p_1 are countably additive; Σ and Σ_1 are σ-fields), we define the Saks metric spaces $\Sigma(p)$ and $\Sigma(p_1)$ as in [9, p. 156].

An equivalence relation is defined on Σ by setting A equal to B if

$$p(A\nabla B) \equiv p((A\cup B) - (A\cap B)) = 0 .$$

If [A] represents the set of all elements of Σ which are equivalent to A under the above equivalence relation, then a distance function d can be defined on the resulting set of equivalence classes by

$$d([A], [E]) = p(A\Delta E)$$

where A and E are arbitrary representatives of the equivalence classes [A] and [E]. The set of all equivalence classes equipped with the above distance function d is a metric space and is denoted by $\Sigma(p)$. $\Sigma_1(p_1)$ is defined in an identical fashion. As shown in [9] the above metric spaces are complete.

By III.7.1 of [9] the closure in $\Sigma(p)$ of B is a σ-field. Consequently B is dense in $\Sigma(p)$. Similarly B_1 is dense in $\Sigma_1(p_1)$. As usual, the norm $\| \ \|$ of an element of $\Sigma(p)$ is defined as the distance from the zero element $[\phi]$.

Lemma 3.7. The function γ defined above establishes an isometric isomorphism, or isometry, between $\Sigma(p)$ and $\Sigma_1(p_1)$.

Proof. Let d_1, $\| \ \|$, denote the distance function and the corresponding norm in $\Sigma_1(p_1)$. For any set $E \in B$

$$\| E \| = d([E], [\phi]) = p(E\Delta\phi)$$

$$= p(E) = p_1(\gamma(E))$$

$$= p_1(\gamma(E)\Delta\phi) = d_1([\gamma(E)], [\phi])$$
$$= \|\gamma(E)\|_1 .$$

Similarly for any set E_1 in B_1 $\|E_1\|_1 = \|\gamma^{-1}(E_1)\|$.

As noted earlier γ maps B onto B_1 and for E, F ϵ B

$$\gamma(E \cup F) = \gamma(E) \cup \gamma(F)$$

and

$$\gamma(E \cap F) = \gamma(E) \cap \gamma(F) .$$

Also if $E \neq F$, then $p(E\Delta F) \neq 0$; thus

$$0 \neq p(E\Delta F) = p_1(\gamma(E\Delta F)) = p_1(\gamma(E)\Delta\gamma(F))$$

which implies that $\gamma(E) \neq \gamma(F)$. Consequently γ is a one-to-one algebraic isomorphism between the dense subsets B and B_1 of $\Sigma(p)$ and $\Sigma_1(p_1)$.

To see that γ is continuous as a mapping from B to B_1, note that if the sequence $(E_n) \subset B \subset \Sigma(p)$ converges to $E \in B$, then

$$\lim_n p(E_n \Delta F) = 0 .$$

Consequently

$$\lim_n p_1(\gamma(E_n)\Delta\gamma(E)) = 0 ,$$

which implies that $\gamma(E_n) \xrightarrow{n} \gamma(E)$ in $\Sigma_1(p_1)$.

We have shown that γ establishes an isometry between B and B_1 when B, B_1 are endowed with the metrics inherited from $\Sigma(p)$ and $\Sigma(p_1)$ respectively. By Theorem I. 6. 17 of [9], since B_1 is dense in $\Sigma_1(p_1)$ and $\Sigma_1(p_1)$ is a complete metric space this is sufficient to imply that γ can be extended to an isometry between $\Sigma(p)$ and $\Sigma_1(p_1)$.

Q.E.D.

In several of the following theorems the proof involves a verification that a function is an isometry between two metric spaces.

As in [9] let $F(S, \Sigma, p, X)$ denote the space of all equivalence classes of functions from S to X with the norm

$$|f| = \inf_{\alpha>0} \{\alpha + p^*\{s: s \in S, |f(s)| > \alpha\}\}$$

where

$$p^*(E) = \inf \{p(F): E \subset F, F \in \Sigma\} .$$

We will refer to elements of $F(S, \Sigma, p, X)$ as if they were functions rather than equivalence classes of functions, with the understanding that two functions which are equal p-a.e. are the same. As is shown in [9], $F(S, \Sigma, p, X)$ is a complete metric space and convergence in $F(S, \Sigma, p, X)$ is equivalent to convergence in p-measure.

The space $TM(S, \Sigma, p, X)$ of all X-valued totally p-measurable function on S is defined to be the closure of the set of Σ-simple functions in $F(S, \Sigma, p, X)$. That is, $TM(S, \Sigma, p, X)$ is the closure in $F(S, \Sigma, p, X)$ of the set of Σ-simple functions

$$\{ \sum_{i=1}^{n} x_i I_{E_i}(\cdot) : x_i \in X,\ E_i \in \Sigma,\ n \in Z\} \quad .$$

The space $TM(S_1, \Sigma_1, p_1, X)$ is defined analogously.

Lemma 3.8. If φ is defined on the Σ-simple functions of $TM(S, \Sigma, p, X)$ by

$$\varphi\left(\sum_{i=1}^{n} x_i I_{E_i}(\cdot)\right) = \sum_{i=1}^{n} x_i I_{\gamma(E_i)}(\cdot)$$

then there is a unique extension of φ to an isometry between $TM(S, \Sigma, p, X)$ and $TM(S_1, \Sigma_1, p_1, X)$.

Proof. Since the Σ_1-simple functions in $TM(S_1, \Sigma_1, p_1, X)$ are dense and $TM(S_1, \Sigma_1, p_1, X)$ is a complete metric space, it suffices to prove that φ establishes an isometry between the Σ-simple functions of $TM(S, \Sigma, p, X)$ and the Σ_1-simple functions of $TM(S_1, \Sigma_1, p_1, X)$.

The linearity of φ for simple functions is clear from the definition. To establish that φ is isometric recall that the norm of a Σ-simple function $\sum_{i=1}^{n} x_i I_{E_i}(\cdot)$ in $TM(S, \Sigma, p, X)$ is defined as

$$\left\| \sum_{i=1}^{n} x_i I_{E_i}(\cdot) \right\| = \inf_{\alpha > 0} \{\alpha + p^*\{s : s \in S_1, \left\| \sum_{i=1}^{n} x_i I_{E_i}(s) \right\| \geq \alpha\}\}.$$

Note that

$$\begin{aligned} \left\| \sum_{i=1}^{n} x_i I_{E_i}(s) \right\| &= 0 \quad \text{if } s \notin E_i,\ i=1,\ldots,n, \\ &= \| x_i \| \quad \text{if } s \notin E_i,\ i=1,\ldots,n \ . \end{aligned}$$

Let $J_\alpha \subset \{1,\ldots,n\}$ denote those indicies i for which $\| x_i \| > \alpha$. Then

$$p^*\{s: s \in S, \| \sum_{i=1}^{n} x_i I_{E_i}(s) \| > \alpha\} = \sum_{i \in J_\alpha} p(E_i) \quad \text{if } J_\alpha \neq \phi ,$$
$$= p(\phi) = 0 \quad \text{if } J_\alpha = \phi \quad .$$

Consequently

$$\{\alpha + p^*\{s: s \in S, \| \sum_{i=1}^{n} x_i I_{E_i}(s) \| > \alpha\}\} = \alpha + \sum_{i \in J_\alpha} p(E_i) \; .$$

Therefore if the x_i, $i=1,\dots,n$, are ordered so that

$$\| x_1 \| \leq \| x_2 \| \leq \dots \leq \| x_n \| \quad ,$$

then

$$\| \sum_{i=1}^{n} x_i I_{E_i}(\cdot) \| = \inf_{\alpha > 0} \{\alpha + \sum_{i \in J_\alpha} p(E_i)\}$$
$$= (\sum_{i=1}^{n} p(E_i)) \wedge (\| x_i \| + \sum_{i=2}^{n} p(E_i)) \wedge \dots \wedge (\| x_{n-1} \| + p(E_n)) \wedge (\| x_n \|) .$$

Since

$$p_1(\gamma(E_i)) = p(E_i) \quad i=1,\dots,n \; ,$$

it follows as above that

$$\| (\sum_{i=1}^{n} x_i I_{E_i}(\cdot)) \| = \| \sum_{i=1}^{n} x_i I_{\gamma(E_i)}(\cdot) \|$$
$$= (\sum_{i=1}^{n} p_1(\gamma(E_i))) \wedge (\| x_i \| + \sum_{i=2}^{n} p_1(\gamma(E_i))) \wedge \dots \wedge (\|x_n\|$$
$$= (\sum_{i=1}^{n} p(E_i)) \wedge (\| x_1 \| + \sum_{1=2}^{n} p(E_i)) \wedge \dots \wedge (\| x_n \|)$$
$$= \| \sum_{i=1}^{n} x_i I_{E_i}(\cdot) \| \quad .$$

Q.E.D.

Lemma 3.9. The function φ defined in the above lemma establishes an isometry between $P_1(p, X)$ and $P_1(p_1, X)$ such that if $\varphi(f)$ is denoted by $\bar{f}$ for each f in $P_1(p, X)$ then

$$\int_E f dp = \int_{\gamma(E)} \bar{f} dp_1$$

for every $E \in \Sigma$.

Proof. Let f be a function belonging to $P_1(p, X)$. By Theorem 3.4 there is a sequence of simple functions (f_n) belonging to $P_1(p, X)$ such that

$$f_n(s) \xrightarrow{n} f(s) \text{ p-a.e. and } \| f_n - f \|_{P_1} \xrightarrow{n} 0 .$$

Each function f_n of the above sequence is of the form

$$f_n = \sum_{i=1}^{K} x_i I_{E_i}(\cdot),\ x_i \in X,\ E_i \in \Sigma,\ i=1,\ldots,K .$$

To see that $\bar{f}_n$ is in $P_1(p_1, X)$ and that $\| f_n \|_{P_1} = \| \bar{f}_n \|_{P_1}$ note that for each $x^* \in X^*$

$$\begin{aligned}\int_S |x^* f_n| dp &= \sum_{i=1}^{K} \int_{E_i} |x^* x_i I_{E_i}(\cdot)| dp = \sum_{i=1}^{K} |x^* x_i| p(E_i) \\ &= \sum_{i=1}^{K} |x^* x_i| p_1(\gamma(E_i)) = \sum_{i=1}^{K} \int_{\gamma(E_i)} |x^* x_i| dp_1 \\ &= \int_{S_1} |x^* \sum_{i=1}^{K} x_i I_{\gamma(E_i)}(\cdot)| dp_1 \\ &= \int_{S_1} |x^* \bar{f}_n| dp_1\end{aligned}$$

and therefore

$$\| f_n \|_{P_1} = \sup_{\| x^* \| \le 1} \int_S |x^* f_n| dp$$

$$= \sup_{\| x^* \| \le 1} \int_{S_1} |x^* \bar{f}_n| dp_1$$

$$= \| \bar{f}_n \|_{P_1} .$$

For each set F in Σ_1

$$\int_F \bar{f}_n dp_1 = \int_F \left(\sum_{i=1}^{n} x_i I_{\gamma(E_i)}(\cdot) \right) dp_1$$

$$= \sum_{i=1}^{K} x_i p_1(F \cap \gamma(E_i))$$

$$= \sum_{i=1}^{K} x_i p(\gamma^{-1}(F) \cap E_i)$$

$$= \int_{\gamma^{-1}(F)} f_n dp .$$

Since

$$\| f_n - f \|_{P_1} \xrightarrow{n} 0$$

implies that the sequence (f_n) is Cauchy in $P_1(p, X)$, an application of Lemma 3.5 shows that

$$\sup_{E \in \Sigma} \| \int_E f_n dp - \int_E f_m dp \|_X \to 0$$

as n, m → ∞. Thus as a consequence of the equality

$$\int_F \bar{f}_n dp_1 = \int_{\gamma^{-1}(F)} f_n dp$$

for all sets F in Σ we have that

$$\sup_{F \in \Sigma} \| \int_F \bar{f}_n dp_1 - \int_F \bar{f} dp_1 \|_X \to 0$$

as $n, m \longrightarrow \infty$. Therefore a second application of Lemma 3.5 shows that the sequence $(\overline{f}_n)$ is Cauchy in $P_1(p_1, X)$.

Since the sequence (f_n) converges to f p-a.e. it also converges in p-measure. Consequently since φ is an isometry between $TM(S, \Sigma, p, X)$ and $TM(S_1, \Sigma_1, p_1, X)$ the sequence $(\overline{f}_n)$ converges to $\overline{f}$ in p_1-measure. Convergence in measure implies weak convergence in measure so the conditions of Theorem 3.5 are met for the sequence $(\overline{f}_n)$. Therefore $\overline{f}$ is in $P_1(p_1, X)$ and

$$\| \overline{f}_n - \overline{f} \|_{P_1} \xrightarrow{n} 0 .$$

The fact that $\int_E f dp = \int_{\gamma(E)} \overline{f} dp_1$ for every set E in Σ follows immediately since

$$\int_E f dp = \lim_n \int_E \overline{f}_n dp = \lim_n \int_{\gamma(E)} \overline{f} dp_1 = \int_{\gamma(E)} dp_1 .$$

To show that φ is an isometric mapping note that since the sequences $(\| f_n - f \|_{P_1})$ and $(\| \overline{f}_n - \overline{f} \|_{P_1})$ both converge to zero as $n \longrightarrow \infty$ we may, by choosing a subsequence if necessary, assume that

$$\| f_n - f \|_{P_1} < \frac{1}{n} \quad \text{and} \quad \| \overline{f}_n - \overline{f} \|_{P_1} < \frac{1}{n}$$

for each $n \in Z$. From these inequalities it follows that

$$\big| \, \| f_n \|_{P_1} - \| f \|_{P_1} \big| < \frac{1}{n} \quad \text{and} \quad \big| \, \| \overline{f}_n \|_{P_1} - \| \overline{f} \|_{P_1} \big| < \frac{1}{n} .$$

Consequently since $\| f_n \|_{P_1} = \| \overline{f}_n \|_{P_1}$ for each $n \in Z$ we see that

$$\| f \|_{P_1} = \| \overline{f} \|_{P_1} .$$

The linearity of φ follows from the obvious linearity for simple functions.

To show that φ is onto $P_1(p_1, X)$, let f' be an element of $P_1(p_1, X)$. By Lemma 3.8 $\varphi^{-1}(f')$ is in $TM(S, \Sigma, p, X)$. To see that it is also an element of $P_1(p, X)$ note that by Theorem 3.4 there is a sequence of simple functions $(f'_n) \subset P_1(p_1, X)$ such that

$$\| f'_n - f' \|_{p_1} \rightarrow 0 \text{ and } (f_n') \rightarrow f' \quad p_1\text{-a.e.} \quad .$$

For each $n \in Z$, f'_n is of the form

$$f'_n = \sum_{i=1}^{K} x_i I_{F_i}(\cdot), \; x_i \in X, \; F_i \in \Sigma_1, \; i=1,\ldots,K \quad .$$

Consequently

$$\varphi^{-1}(f'_n) = \sum_{i=1}^{K} x_i I_{\gamma^{-1}(F_i)}(\cdot)$$

is clearly a simple function belonging to $P_1(p, X)$. Since the sequence (f'_n) is Cauchy in $P_1(p_1, X)$, it follows as above that $(\varphi^{-1}(f'_n))$ is Cauchy in $P_1(p, X)$ and

$$\varphi^{-1}(f'_n) \xrightarrow{n} \varphi^{-1}(f')$$

in p-measure. Therefore by Theorem 3.6 $\varphi^{-1}(f')$ is in $P_1(p, X)$.

Q.E.D.

The next theorem is the *raison d'être* of the last four lemmas. It enables us to translate an amart defined on an arbitrary space S to an amart defined on the Stone space S_1 where any finitely additive measure is also countably additive. This fact will prove to be of major importance in proving Theorem 3.16.

Theorem 3.10. Let $\{f_\alpha : \alpha \in \Lambda\} \subset P_1(p, X)$ be an amart for the increasing net of sigma-fields $\{\Sigma_\alpha : \alpha \in \Lambda\}$. If $\varphi(f_\alpha)$ is denoted by $\overline{f}_\alpha$ for each $\alpha \in \Lambda$, then

$$\{\overline{f}_\alpha : \alpha \in \Lambda\} \subset P_1(p_1, X)$$

is an amart for

$$\{\sigma(\gamma(\Sigma_\alpha)) : \alpha \in \Lambda\} \equiv \{\overline{\Sigma}_\alpha : \alpha \in \Lambda\} .$$

Proof. Since f_α is adapted to Σ_α for each $\alpha \in \Lambda$, there is a sequence of Σ_α-simple functions converging in p-measure to f_α. An application of Lemma 3.8 provides a sequence of $\overline{\Sigma}_\alpha$-simple functions which converges in p_1-measure to $\overline{f}_\alpha$. Consequently $\overline{f}_\alpha$ is adapted to $\overline{\Sigma}_\alpha$ for each α in Λ.

Let T' denote the set of stopping times from S_1 to Λ. That is, T' is the set of all functions $\tau' : S_1 \to \Lambda$ with τ' taking only finitely many values and such that $\{\tau' = \alpha\} \in \overline{\Sigma}_\alpha$ for all $\alpha \in \Lambda$.

It must be shown that the net $(\int_{S_1} \overline{f}_{\tau'}\, dp_1 : \tau' \in T')$ converges in X.

Choose $\tau' \in T'$, and let $\alpha_1, \ldots, \alpha_n$ denote the values assumed by τ'. Then

$$\begin{aligned} \int_{S_1} \overline{f}_{\tau'}\, d_{p_1} &= \sum_{i=1}^{n} \int_{\{\tau'=\alpha_i\}} \overline{f}_{\alpha_i}\, dp \\ &= \sum_{i=1}^{n} \int_{\gamma^{-1}\{\tau'=\alpha_i\}} f_{\alpha_i}\, dp \\ &= \int_S f_\tau\, dp \end{aligned}$$

where τ is defined by

$$\tau(\cdot) = \sum_{i=1}^{n} \alpha_i I_{\gamma^{-1}\{\tau'=\alpha_i\}}(\cdot)$$

and is consequently a stopping time in T.

We thus see that for each stopping time $\tau' \in T'$ there is a corresponding stopping time $\tau \in T$ such that

$$\int_S f_\tau dp = \int_{S_1} \bar{f}_{\tau'} dp_1 \quad .$$

Since $\{f_\alpha, \Sigma_\alpha : \alpha \in \Lambda\}$ is an amart in $P_1(p, X)$, for each $\varepsilon > 0$ there is a stopping time $\tau_\varepsilon \in T$ such that if $\tau_1, \tau_2 \in T$ are greater than or equal to τ_ε, then

$$\| \int_S f_{\tau_1} dp - \int_S f_{\tau_2} dp \|_X < \varepsilon \ .$$

Since τ_ε assumes only finitely many values in Λ, τ'_ε can be chosen so that the corresponding stopping time in T is greater than or equal to τ_ε. It follows that if $\tau_1', \tau_2' \in T'$ are greater than or equal to τ_ε',

$$\| \int_{S_1} f_{\tau_1'} dp_1 - \int_{S_1} f_{\tau_2'} dp_1 \|_X < \varepsilon \quad .$$

This proves that the net $(\int_{S_1} \bar{f}_{\tau'} dp_1 : \tau \in T')$ converges in X and consequently that $\{\bar{f}_\alpha, \bar{\Sigma}_\alpha : \alpha \in \Lambda\}$ is an amart.

Q.E.D.

The following theorem due to Rieffel [19] provides the key for extending the martingale convergence theorem developed by Uhl in [21] for random variables taking values in a Banach space to an amart convergence theorem for random variables taking values in a locally convex topological vector space V.

Theorem 3.11. [19, p. 482] Let V be a locally convex topological vector space and K be a compact subset of V. Let K' be the closed circled convex extension of K and let W be the linear manifold spanned by K'. Then a norm can be defined on W such that on K the norm topology coincides with the original topology if and only if the original topology on K has a countable base.

If f is a measurable function taking values in a Banach space X, and if $E \in \Sigma$, then the essential range of f restricted to E denoted by $er_E(f)$ is defined to be the set of all $b \in X$ such that for every $\varepsilon > 0$ the measure of the set

$$\{x \in E: \| f(x) - b \| < \varepsilon\}$$

is strictly positive. The set $er_E(f)$ is clearly a closed subset of X and is contained in the closure of the range of f restricted to E.

For a Bochner integrable function f defined on (S, Σ, p) and taking values in a Banach space X, define the set function $\mu_f: \Sigma \to X$ by

$$\mu_f(E) = \int_E f dp \qquad \text{for} \quad E \in \Sigma .$$

For each $E \in \Sigma$ the average range of μ_f on E is defined by

$$A_E(\mu_f) = \{\mu_f(F)/p(F): F \in \Sigma, F \subseteq E, p(F) > 0\} .$$

Proposition 3.12. [19, p. 471] If f is a Bochner integrable function defined on (S, Σ, p) and taking values in a Banach space X, then for each $E \in \Sigma$ the closure of $A_E(\mu_f)$, denoted by $\bar{A}_E(\mu_f)$, contains $er_E(f)$.

Proposition 3.13. [19, p. 470] If f is a measurable function from S to X and E is an element of Σ, then the set $\{x \in E: f(x) \notin er_E(f)\}$ is p-null.

Theorem 3.14. [22, p. 674] Let $F: \Sigma \longrightarrow X$ be finitely additive. Then $F(\Sigma)$ is conditionally weakly compact if and only if there is a finitely additive non-negative measure μ on Σ such that F is μ-continuous.

The following adaptation of the Radon-Nikodým theorem due to Uhl [23] will be needed in the next proof.

Theorem 3.15. [23, p. 371] Let Σ_0 be a subfield of Σ such that $\sigma(\Sigma_0) = \Sigma$. If F is a countably additive, Banach-valued vector measure defined on Σ_0, then F admits the representation

$$F(E) = \int_E f dp$$

for $E \in \Sigma_0$, where the integral is that of Pettis, if and only if:

(a) F has a bounded range

(b) F is p-continuous; i.e. $\lim_{p(E)\to 0} \| F(E) \| = 0$

(c) Given $\varepsilon > 0$ there exists a weakly compact convex subset $K \subset X$ such that for any $\delta > 0$ there exists $E_0 \in \Sigma_0$ with $p(S \backslash E_0) < \varepsilon$ and $F(E) \in p(E) \cdot K + \delta U$ for all $E \in \Sigma_0$, $E \in \Sigma_0$ where U is the closed unit ball of X.

In [21] Uhl gave conditions similar to those of the following theorem which insured strong convergence in measure of a martingale consisting of strongly measurable Pettis integrable random variables taking values in a Banach space. By applying Theorem 3.11 we are able to adapt the technique of proof used by Uhl to obtain an amart convergence theorem for strongly measurable Pettis integrable random variables which take values in a locally convex topological vector space V.

Theorem 3.16. Let $\{f_\alpha, \Sigma_\alpha: \alpha \in \Lambda\}$ be an amart of strongly measurable, Pettis integrable random variables taking values in a locally convex topological vector space V. If $\{f_\alpha: \alpha \in \Lambda\}$ satisfies the following conditions, it converges weakly in measure to a Pettis integrable function f_∞.

(a) $\text{Sup}\ \{\int |x^* f_\alpha| dp: \|x^*\| \leq 1,\ \alpha \in \Lambda\} < \infty$.

(b) $\{\int_E f_\alpha dp: E \in \Sigma_\alpha,\ \alpha \in \Lambda\}$ is contained in a weakly compact subset J of V such that the relative topology on J has a countable base.

(c) For every $\varepsilon > 0$ there exists a weakly compact set $K \subset V$ such that for any $\delta > 0$ there exists an $\alpha_0 \in \Lambda$ and an $E_0 \in \Sigma_{\alpha_0}$ with $p(S \backslash E_0) < \varepsilon$ such that if $\alpha \geq \alpha_0$, then

$$\int_E f_\alpha dp \in p(E) \cdot K + \delta U$$

for every set E in Σ_0 which is a subset of E_0 . U denotes the unit ball in V.

Proof. Let J' denote the closed circled convex extension of J and let W be the linear manifold spanned by J'. Applying Theorem 3.11, a norm $\|\cdot\|$ can be defined on W which generates a topology equivalent to the weak topology on V restricted to W. Let W_o be the completion of W with respect to this norm. Hence W_o is a Banach Space.

For each $\alpha \in \Lambda$ an application of Proposition 3.12 shows

$$er_S(f_\alpha) \subseteq \overline{A}_S(\mu_f) = cl\{\int_E f_\alpha dp/p(E): E \in \Sigma_\alpha\} \subseteq W_o \ .$$

Since, by Proposition 3.13,

$$\{s \in S: f_\alpha(s) \notin er_S(f_\alpha)\} \equiv N_\alpha$$

is a p-null set, we see that the range of f_α is almost everywhere in W_o .

If $B = \bigcup_{\alpha\in\Lambda} \Sigma_\alpha$ then we may assume that $\Sigma = \sigma(B)$. As in Theorem 3.2 there is an isomorphism γ from the field B onto the field B_1 of all open, closed subsets of some compact Hausdorff space S_1.

By assumption

$$\sup \{\int |x^* f_\alpha| dp: \alpha \in \Lambda, ||x^*|| \leq 1, x^* \in V^*\} < \infty \ .$$

Since the norm topology on W_o is equivalent to the restriction of the weak topology of V to W_o, the above inequality remains valid if the $\{x^*\}$ are taken to be elements of W_o^*. Using the notation for the Pettis norm defined previously and noting that

$$\{f_\alpha: \alpha \in \Lambda\} \subset P_1(p, W_o) \quad ,$$

the above inequality becomes

$$\sup_{\alpha\in\Lambda} || f_\alpha ||_{P_1} < \infty \quad .$$

Define p_1 on B_1 by

$$p_1(\gamma(E)) = p(E) \quad \text{for } E \in B .$$

Applying Theorem 3.3, p_1 is countably additive on B_1 and can be uniquely extended to the σ-field Σ_1 generated by B_1. Let p_1 denote this extension. By Lemma 3.9 the map φ defined on the indicator functions of sets in B by $\varphi(I_E) = I_{\gamma(E)}$ establishes an isometry between $P_1(p, W_o)$ and $P_1(p_1, W_o)$ such that for each $\alpha \in \Lambda$ and $E \in \Sigma$

$$\int_E f_\alpha dp = \int_{\gamma(E)} (f_\alpha) dp_1 \quad .$$

An application of Theorem 3.10 shows that if $\varphi(f_\alpha)$ is denoted by $\overline{f}_\alpha$ then $\{\overline{f}_\alpha, \overline{\Sigma}_\alpha : \alpha \in \Lambda\}$ is an amart, where $\overline{\Sigma}_\alpha = \sigma(\gamma(\Sigma_\alpha))$.

For each $E \in B$ $\lim_{\tau \in T} (\int_E f_\tau dp)$ exists in the norm of W_o. To show this let $\varepsilon > 0$ be given; if $E \in \Sigma_{\alpha_1}$ choose $\alpha_2 > \alpha_1$ such that if τ_1 and τ_2 are in T and $\tau_1, \tau_2 \geq \alpha_2$ then

$$\| \int f_{\tau_1} dp - \int f_{\tau_2} dp \|_{W_o} < \varepsilon \quad .$$

Since τ_1 and τ_2 assume only finitely many values in Λ there is an element $\alpha_o \in \Lambda$ such that $\alpha_o \geq \max \{\tau_1, \tau_2\}$. Define stopping times τ_1' and τ_2' as follows.

$$\begin{aligned} \tau_1'(s) &= \tau_1(s) && \text{if } s \in E \\ &= \alpha_o && \text{if } s \notin E , \\ \tau_2'(s) &= \tau_2(s) && \text{if } s \in E \\ &= \alpha_o && \text{if } s \notin E . \end{aligned}$$

To see that τ_1' is a stopping time, let $\{\gamma_1, \ldots, \gamma_n\}$ denote the elements of Λ in the range of τ_1. Then the range of τ_1' is $\{\gamma_1, \ldots, \gamma_n, \alpha_o\}$, and

$$\{\tau_1' = \gamma_i\} = E \cap \{\tau_1 = \gamma_i\} \in \Sigma_{\gamma_i} \quad \text{for } i = 1, \ldots, n$$

and

$$\{\tau_1' = \alpha_o\} = E^c \in \Sigma_{\alpha_o} .$$

Similarly τ_2' is a stopping time.

Observe that since $\tau_1', \tau_2' > \alpha_2$,

$$\|\textstyle\int_E f_{\tau_1} dp - \int_E f_{\tau_2} dp\| = \|\int_E f_{\tau_1} dp + \int_{E^c} f_{\alpha_o} dp - \int_{E^c} f_{\alpha_o} dp - \int_E f_{\tau_2} dp\|_{W_o}$$

$$= \|\textstyle\int_S f_{\tau_1'} dp - \int_S f_{t_2'} dp\|_{W_o}$$

$$< \varepsilon .$$

Consequently $\lim\limits_{\tau \in T} (\int_E f_\tau dp)$ exists in the norm of W_o.

For each $E \in B$ define

$$F(E) = \lim_{\tau \in T} \int_E f_\tau dp .$$

F is finitely additive on B and by condition (b), F(B) is contained in a weakly compact subset of W_o. As a result of Theorem 3.14 there exists a finitely additive, non-negative measure v on B such that

$$\lim_{v(E) \to 0} \|F(E)\| = 0 .$$

That is, F is v-continuous. Define v_1 on B_1 by

$$v_1(\gamma(E)) = v(E) \quad \text{for } E \in B .$$

Then by Theorem 3.3 v_1 is countably additive and has a unique extension to a countably additive measure on Σ_1, the σ-field generated by B_1. If F_1 is defined on B_1 by

$$F_1(\gamma(E)) = F(E) \quad \text{for } E \in B ,$$

then F_1 is v_1-continuous and countably additive on B_1. To see that F_1 is countably additive on B_1 note that if $\{E_i : i \in Z\}$ is a disjoint collection of sets in B_1 such that $\cup E_i \in B_1$, then since $\cup E_i$ is closed in S_1, it is compact. Hence there must exist a finite subcover of the countable open cover $\{E_i : i \in Z\}$ of $\cup E_i$. Therefore $E_i = \phi$ for all but finitely many i. Hence F_1 is countably additive on B_1.

If $\{E_i : i \in Z\}$ is a disjoint sequence of sets in B_1, then since v_1 is countably additive on B_1, $V_1(E_i) \to 0$ as $i \to \infty$. Consequently, since F_1 is v_1-continuous, $F_1(E_i) \to 0$ as $i \to \infty$. By Theorem 2.2 of Brooks and Dinculeanu [3] this implies that F_1 has a unique extension to a countably additive W_0-valued measure on Σ_1.

As a consequence of condition (c), for any $\varepsilon > 0$ there exists a weakly compact set $K \subset W_0$ such that if $\delta > 0$ is given, there is an $E_0 \in B$ such that $p(S \setminus E_0) < \varepsilon$ and

$$F(E) \in p(E) \cdot K + \delta U$$

for every set E in B which is a subset of E_0. Consequently for every set E in B_1 such that $E \subset \gamma(E)$

$$F_1(E) \in p_1(E) \cdot K + \delta U$$

and $p_1(S \backslash \gamma(E_0)) < \varepsilon$.

Define $\lambda_1 = p_1 + v_1$. Then if $E \in B_1$ and $\lambda_1(E) \neq 0$ we have

$$p_1(E) = \lambda_1(E) \cdot p_1(E)/\lambda_1(E) \ .$$

Consequently if $E \in B_1$ is such that $E \subset \gamma(E_0)$, then

$$F_1(E) \in \lambda_1(E) \cdot K_1 + \delta U$$

where $K_1 = \{\beta \cdot x\colon 0 \le \beta \le 1,\ x \in K\}$.

Since K_1 is weakly compact the vector measure F_1 satisfies the conditions of Theorem 3.15. Consequently there is a $g \in P_1(\lambda_1, W_0)$ such that

$$F_1(E) = \int_E g\,d\lambda_1 \quad \text{for } E \in B_1 \quad .$$

Since $(S_1, \Sigma_1, \lambda_1)$ is a finite positive measure space and p_1 is clearly λ_1-continuous, an application of the Radon-Nikodým Theorem for scalar measures provides a Radon-derivative $h = dp_1/d\lambda_1$. Let h $= E[h|\overline{\Sigma}_\alpha]$, the conditional expectation of h given $\overline{\Sigma}_\alpha$. Since h is in $L_1(\lambda_1)$ it is well known (see Doob [8]) that $(h_\alpha, \overline{\Sigma}_\alpha, \alpha \in A)$ is a uniformly integrable martingale and hence $(h_\alpha\colon \alpha \in A)$ converges in $L_1(\lambda_1)$ to h. Since

$$\int_E \overline{f}_\alpha dp_1 = \int_E \overline{f}_\alpha h\,d\lambda_1 = \int_E \overline{f}_\alpha h_\alpha d\lambda_1$$

for all $E \in \overline{\Sigma}_\alpha$, it follows that for $E \in B_1$,

$$\begin{aligned}\lim_{\tau'\in T'} \int_E \bar{f}_{\tau'} h_{\tau'} d_{\lambda_1} &= \lim_{\tau'\in T'} \int_E \bar{f}_{\tau'} dp_1 \\ &= F_1(E) \\ &= \int_E g d\lambda_1 \quad .\end{aligned}$$

For each $x^* \in W_0^*$ $\lim_{\tau'\in T'} \int x^*\bar{f}_{\tau'} h_{\tau'} d_{\lambda_1}$ exists and

$$\begin{aligned}\sup_{\alpha\in\Lambda} \int |x^*\bar{f}_\alpha h_\alpha| d_{\lambda_1} &= \sup_{\alpha\in\Lambda} \int |x^*\bar{f}_\alpha| dp_1 \\ &= \sup_{\alpha\in\Lambda} \int |x^* f_\alpha| dp \\ &< \infty \ .\end{aligned}$$

Consequently $\{x^*\bar{f}_\alpha h_\alpha, \bar{\Sigma}_\alpha, \alpha \in \Lambda\}$ is a scalar-valued L_1-bounded amart. By Theorem 2.9 $\{x^*f_\alpha h_\alpha: \alpha \in \Lambda\}$ converges in λ_1-measure to a function in $L_1(\Sigma_1, \lambda_1)$, which we shall denote by g_{x^*}, and

$$\lim_{\tau'\in T'} \int_E x^*\bar{f}_{\tau'} h_{\tau'} d_{\lambda_1} = \int_E g_{x^*} d_{\lambda_1}$$

for every $E \in B_1$. From this it follows that

$$\int_E x^* g d_{\lambda_1} = \lim_{\tau'\in T'} \int_E x^*\bar{f}_{\tau'} h_{\tau'} d_{\lambda_1} = \int_E g_{x^*} d_\lambda ,$$

for every $E \in \Sigma_1$ and consequently

$$g_{x^*} = x^* g \quad \lambda_1 \text{ - a.e.}$$

for all $x^* \in W_0^*$. We have thus shown that $\{x^*\bar{f}_\alpha h_\alpha: \alpha \in \Lambda\}$ converges in λ_1-measure to x^*g for each $x^* \in W_0^*$. By definition this implies that $\{\bar{f}_\alpha h_\alpha: \alpha \in \Lambda\}$ converges weakly in λ_1-measure to g. Consequently, since $\lambda_1 = p_1 + v_1$, $\{\bar{f}_\alpha h_\alpha: \alpha \in \Lambda\}$ converges to g weakly in p_1-measure. Since

$h = dp_1/d\lambda_1$ it follows that $p_1(\{h=0\}) = \int_{\{h=0\}} hd_{\lambda_1} = 0$; therefore $\{\overline{f}_\alpha h_\alpha/h \colon \alpha \in \Lambda\}$ converges to g/h weakly in p_1-measure. Since $\{h_\alpha \colon \alpha \in \Lambda\}$ converges to h in λ_1-measure it also converges to h in p_1-measure. Consequently $\{h_\alpha/h \colon \alpha \in \Lambda\}$ converges to 1 in p_1-measure. We therefore have

$$\overline{f}_\alpha \xrightarrow{\alpha} g/h$$

weakly in p_1-measure.

Let $S_o = \{h \neq 0\}$, Then $p_1(S_1 \backslash S_o) = 0$ and $\lambda_1(E \cap S_o) \to 0$ as $p_1(E) \to 0$. To see this note that $p_1(E) = \int_E hd_{\lambda_1}$; therefore if $p_1(E) \to 0$, it follows that $\int_{\{E \cap S_o\}} hd_{\lambda_1} \to 0$, and since h is positive, $\lambda_1(E \cap S_o) \to 0$. If $\tilde{\lambda}$ is defined by

$$\tilde{\lambda}(E) = \lambda_1(E \cap S_o)$$

for $E \in \Sigma_1$, then $d\tilde{\lambda}/dp_1 = \frac{1}{h}$. Therefore since $\lambda_1(E) = \tilde{\lambda}(E)$ for $E \in S_o$ we have

$$\int_E gd_{\lambda 1} = \int_E g/h \, dp_1$$

for $E \subset S_o$, $E \in \Sigma_1$. Consequently $g/h \in P_1(p_1, W_o)$. If $\varphi^{-1}(g/h)$ is denoted by f_∞ then by Lemma 3.9 $f_\infty \in P(p, W\ )$ and

$$\int_E f_\infty dp = \int_{(E)} g/h \, dp_1$$

for all $E \in \Sigma$. Also by Lemma 3.9, since

$$\overline{f}_\alpha \xrightarrow{\alpha} g/h \text{ weakly in } p_1\text{-measure,}$$

it follows that

$$f_\alpha \xrightarrow{\alpha} f_\infty \text{ weakly in p-measure.}$$

Q.E.D.

Corollary 3.17. In Theorem 3.16 if Λ is taken to be the positive intergers Z, then $\{f_n : n \in Z\}$ converges weakly a.e. to a Pettis integrable function f_∞.

Proof. The only change required in the proof of the theorem is to use Theorem 2 of Austin-Edgar-Tulcea [1] rather than Theorem 2.10 of Edgar and Sucheston [10] to conclude that $\{x^*\overline{f}_n h_n : n \in Z\}$ converges λ_1-a.e. rather than in λ_1-measure.

Q.E.D.

Corollary 3.18. If V is a Banach space, then the condition that J have a countable base can be removed.

Proof. The only use of this condition was to insure that a norm could be defined on the linear manifold spanned by the closed, circled convex extension of J which generated a topology equivalent to the weak topology on V.

Q.E.D.

The following example derived from an argument in [18] illustrates that a vector-valued amart may satisfy the conditions of Theorem 3.16 (or Theorem 2.8) and thus converge in the weak topology and yet not converge in the strong topology of the vector space.

Example 3.19. Let V be the space ℓ_∞ of all bounded sequences $x = (\alpha_n)$ of complex numbers. The space ℓ_∞ is a Banach space when the norm of an element $x \in \ell_\infty$ is defined by

$$\| x \| = \sup_n |\alpha_n| \quad .$$

For each $n \in Z$ define e_n to be the element of ℓ_∞ whose $n^{\underline{th}}$ term is one and the rest zero. Let $S = [0, 1)$ and for each n define

$$A_n^i = [i/_2n,\ i+1/_2n) \quad i=0,\dots,\ 2^n-1 .$$

Denote by Σ_n the field generated by $\{A_n^i\colon i=0,\dots,\ 2^n-1\}$. Define f_n inductively by setting

$$f_1(s) = e_1 \quad s \in [0, 1)$$

and

$$f_n(s) = f_{n-1}(s) + \sum_{i=0}^{2^n-1} (-1)^{i+n} e_n I_{A_n^i}(s) \quad .$$

If p is Lebesgue measure on $[0, 1)$, then $(f_n, \Sigma_n, n \in Z)$ is a strongly measurable, strongly integrable amart. In fact it is a martingale. Note that

$$\int_{A_{n-1}^j} f_n dp = \int_{A_n^{2j}} f_n dp + \int_{A_n^{2j+1}} f_n dp$$

$$= \int_{A_n^{2j}} f_{n-1} dp + \int_{A_n^{2j}} \Big(\sum_{i=0}^{2^n-1} (-1)^{i+n} e_n I_{A_n^i}\Big) dp$$

$$+ \int_{A_n^{2j+1}} f_{n-1} dp + \int_{A_n^{2j+1}} \Big(\sum_{i=0}^{2^n-1} (-1)^{i+n} e_n I_{A_n^i}\Big) dp$$

$$= \int_{A_n^{2j}} f_{n-1} dp + \int_{A_n^{2j+1}} f_{n-1} dp + \Big(\frac{1}{2^n}\Big)(-1)^{2j+n} e_n$$

$$+ \Big(\frac{1}{2^n}\Big)(-1)^{2j+1+n} e_n$$

$$= \int_{A_{n-1}^j} f_{n-1} dp \ .$$

Consequently $\int_B f_n dp = \int_B f_{n-1} dp$ for any $B \in \Sigma_{n-1}$. Since $\| f_n(s) \|_\infty = 1$ for each $n \in Z$, $\| f_\tau(s) \|_\infty = 1$ for any stopping time $\tau \in T$. Thus $(f_n, \Sigma_n, n \in Z)$ is a uniformly integrable amart. However the sequence (f_n) does not converge in the norm topology of ℓ_∞ since the sequence $(f_n(s))$ is not Cauchy for any $s \in [0, 1)$. Theorem 3.16, and similarly Theorem 2.8, do not apply when ℓ_∞ is given its norm topology since the set $\{\int_E f_n dp: E \in \Sigma_n, n \in Z\}$ is contained in the closed unit ball U of ℓ_∞ which is not weakly compact in this case since ℓ_∞ is not reflexive. However if ℓ_∞ is given the weak topology generated by the duality $\sigma(\ell_\infty, \ell_1)$ then the unit ball U is compact. The above topology on ℓ_∞ is the topology for which the sub-basic open sets about a point $x \in \ell_\infty$ are of the form $\{y \in \ell_\infty: \langle Z, y-x \rangle < \varepsilon, \varepsilon > 0, Z \in \ell_1\}$ where, for $x = (\alpha_n) \in \ell_\infty$ and $Z = (\gamma_n) \in \ell_1$,

$$\langle Z, x \rangle = \sum_{n \geq 1} (\alpha_n \gamma_n) \quad .$$

Since ℓ_∞ is the continuous dual of ℓ_1, the above topology is the weak* topology and compactness of the unit ball U in this topology follows from the Banach-Alaoglu Theorem [9, p. 424]. The unit ball U is easily seen to have a countable base in this topology. Part (c) of Theorem 3.16 is also clearly satisfied. Consequently Corollary 3.17 implies that the sequence (f_n) converges a.e. in the weak * topology of ℓ_∞.

CHAPTER IV
ASYMPTOTIC MARTINGALES INTEGRABLE WITH RESPECT TO A VECTOR MEASURE

In [13] Lewis developed a theory of integration with respect to a measure which takes values in a locally convex linear topological space V. He developed the integration theory chiefly through the study of the P-semi-variation of such a measure where P is a semi-norm on V. The P-semi-variation as defined in Lewis [13] is basically an adaptation to the case of locally convex linear topological spaces of the concept of semi-variation found in Dinculeanu [7].

In this chapter a theory of amarts, or asymptotic martingales, which are integrable in the sense of the integral defined by Lewis [13] is developed along the lines of the corresponding theory of scalar valued amarts developed in [10] by Edgar and Sucheston and in [5] by Chacon.

Throughout this chapter (S, Σ) will denote a measurable space; that is, S is an arbitrary set and Σ is a σ-field of subsets of S.

A continuous semi-norm P defined on V is a continuous function from V into the real numbers such that

$$0 \le P(x) < +\infty ,$$

$$P(\lambda x) = |\lambda| P(x) ,$$

and

$$P(x+y) \le P(x) + P(y)$$

for any $x, y \in V$ and complex number λ. If $x^* \in V^*$ and P is a semi-norm defined on V then $x^* \leq P$ means that $|x^*x| \leq P(x)$ for all $x \in V$. A set function $\mu\colon \Sigma \longrightarrow V$ is weakly countably additive if for every $x^* \in V^*$, the set function $x^*\mu$ is countably additive. Pettis [17] proved that if μ is weakly countably additive, then it is countably additive. If μ is countably additive and $E = \bigcup_{i\geq 1} E_i$, where $\{E_i\colon i \in Z\}$ is a disjoint collection of elements of Σ, then

$$\mu(E) = \sum_{i\geq 1} \mu(E_i) = \sum_{i\geq 1} \mu(E_{\pi(i)})$$

where π is any permutation of the positive integers. Hence if μ is countably additive, then $\sum_{i\geq 1} \mu(E_i)$ is unconditionally convergent.

Throughout this chapter μ will denote a countably additive V-valued measure defined on Σ. If P is a semi-norm on V, then the P-semi-variation of μ, denoted by $\|\mu\|_P$, is defined as in [13] as the function from Σ into the positive real numbers by

$$\|\mu\|_P(E) = \sup \{|x^*\mu|(E)\colon x^* \leq P\}.$$

The expression $|x^*\mu|(E)$ denotes the total variation of the scalar measure $x^*\mu$ over E and is defined by

$$|x^*\mu|(E) = \sup \sum_{i=1}^{n} |x^*\mu(E_i)| ,$$

where the sup is taken over all finite collections $\{E_i\colon i=1,\ldots,n\}$ of disjoint subsets of E in Σ.

For proofs of the next two results concerning basic properties of the P-semi-variation of μ see Lewis [13].

Proposition 4.1. [13, p. 158]. If μ is a countably additive measure then $\|\mu\|_P(\cdot)$ is monotone, countably subadditive and

$$P[\mu(E)] \le \|\mu\|_P(E) \le 4 \sup \{P[\mu(F)]: F \subseteq E, F \in \Sigma\} .$$

Theorem 4.2. [13, p. 158]. If μ is a measure, P a continuous semi-norm and $\{E_n: n \in Z\}$ a convergent sequence of sets of Σ, then

$$\|\mu\|_P(\lim_n E_n) = \lim_n \|\mu\|_P(E_n) .$$

A μ-null set is a subset of a set $E \in \Sigma$ such that $\|\mu\|_P(E) = 0$ for every continuous semi-norm P. The statement that $f = g$ μ-almost everywhere (μ-a.e.) means that the set $\{s: s \in S, f(s) \ne g(s)\}$ is a μ-null set. A function $f: S \rightarrow \mathbb{R}$ is said to be a μ-simple function if it is μ-almost everywhere equal to a function of the form $\sum_{i=1}^{n} c_i I_{E_i}(\cdot)$, where $\{E_i: i=1,\ldots,n\} \subset \Sigma$, $\{c_i: i=1,\ldots,n\} \subset \mathbb{R}$, $\bigcup_{i=1}^{m} E_i = S$, and $I_E(\cdot)$ is the indicator function of E. A function $f: S \rightarrow \mathbb{R}$ is μ-measurable if there exists a sequence of μ-simple functions which converge to f μ-almost everywhere. As in the previous chapters a μ-measurable function will be referred to as a random variable.

In this chapter we use the integration theory developed by Lewis in [13]. A function $f\colon S \longrightarrow \mathbb{R}$ is μ-integrable if f is $x^*\mu$-integrable for $x^* \in V^*$ and if for each E in Σ there is an element of V, denoted by $\int_E f d\mu$, such that $x^*(\int_E f d\mu) = \int_E f d(x^*\mu)$ for each x^* in V^*. The fact that the topology on V is Hausdorff insures that this integral is well-defined. The linearity of this integral follows from the linearity of the corresponding scalar integral for each x^* in V^*. Every measurable simple function $\sum_{i=1}^{n} a_i I_{E_i}(\cdot)$ is μ-integrable and

$$\int_E \left(\sum_{i=1}^{n} a_i I_{E_i}\right) d\mu = \sum_{i=1}^{n} a_i \mu(E \cap E_i)$$

for each $E \in \Sigma$. If f is bounded and μ-integrable, then

$$P[\int_E f d\mu] \le \|\mu\|_P(E) \sup\{f(s)\colon s \in S\}$$

for each $E \in \Sigma$. For an elaboration of the properties of this integral see [13].

Theorem 4.3. [13, p. 160] (a) If f is μ-integrable, then the set function on Σ defined by $\Phi(E) = \int_E d\mu$ is a countably additive measure,

$$\|\Phi\|_P(E) = \sup\{\int_E |f| d|x^*\mu| \colon x^* \le P\},$$

and

$$\lim_{\|\mu\|(E)_P \to 0} \|\Phi\|_P(E) = 0$$

for each continuous semi-norm P.

(b) Let (f_n) be a sequence of μ-integrable functions which converge pointwise to f on S and g be a μ-integrable function such that $|f_n| \leq |g|$ for each n. Then f is μ-integrable if V is sequentially complete, that is if Cauchy sequences in V converge. If f is μ-integrable then

$$\int_E f d\mu = \lim_n \int_E f_n d\mu$$

uniformly with respect to $E \in \Sigma$.

The definition of an amart is analogus to the definition found in chapter one. If (f_n) is a sequence of random variables adapted to an increasing sequence of σ-fields (Σ_n) then $(f_n, \Sigma_n, n \in Z)$ is an amart if and only if

$$\sup \{\int |f_n| d|x^*\mu| : x^* \leq P\} < \infty$$

for each $n \in Z$ and each continuous semi-norm P, and

$$\lim_{\tau \in T} \int f_\tau d\mu$$

exists in V. The existence of this limit means that there is a vector $x \in V$ such that for each $\varepsilon > 0$ and each continuous semi-norm P there is a $\sigma \in T$ such that if $\tau \geq \sigma$ then

$$\sup \{|x^*(\int f_\tau d\mu - x)| : x^* \leq P\} < \varepsilon .$$

In [4] Chacon and Sucheston proved an adaptation of Doob's "maximal" lemma for the case of a Banach-valued measure. The next lemma extends this result to the case of a measure taking values in a locally convex linear topological space V. It will be of major importance in the proofs of subsequent convergence theorems.

Lemma 4.4. Let (f_n) be a sequence of μ-integrable functions adapted to an increasing sequence of sub-σ-fields (Σ_n) of Σ. If

$$\sup \{\| \int f_\tau d\mu \|_P : \tau \in T\} < \infty$$

for each continuous semi-norm P then for $\lambda > 0$

$$\| \mu \|_P(\{s \in S: \sup \{|f_n(s)|\} > \lambda\})$$
$$\leq \frac{1}{\lambda}(\sup \{\int |f_\tau| d|x^*\mu| : \tau \in T, x^* \leq P\})$$

for each continuous semi-norm P.

Proof. For each positive integer m define

$$A_m = \{s \in S: \sup \{|f_n(s)|: n \leq m\} > \lambda\}$$

and define σ: $S \to Z$ by

$$\begin{aligned} \sigma(s) &= \min \{n: n \leq m, |f_n(s)| > \lambda\} && \text{for } s \in A_m \\ &= m && \text{for } s \notin A_m \end{aligned}$$

Since σ is clearly bounded and the set $\{\sigma = n\}$ is in Σ_n for all $n \in Z$, σ is a bounded stopping time, i.e. $\sigma \in T$. Therefore for each continuous semi-norm P

$$\begin{aligned} \sup \{\int |f_\tau| d|x^*\mu| : \tau \in T, x^* \leq P\} &\geq \sup \{\int |f_\sigma| d|x^*\mu| : x^* \leq P\} \\ &\geq \sup \{\int_{A_m} |f_\sigma| d|x^*\mu| : x^* \leq P\}. \end{aligned}$$

Since $f_\sigma(s) > \lambda$ if $s \in A_m$, the last expression above is greater than or equal to $\lambda \| \mu \|_P(A_m)$. Thus

$$\|\mu\|_P(A_m) \le \frac{1}{\lambda} \sup \{\int |f_\tau d|x^*\mu| : \tau \in T,\ x^* \le P\}$$

for every integer m. Since

$$\{s \in S: \sup_n \{|f_n(s)|\} > \lambda\} = \lim_n A_n$$

by Theorem 4.2

$$\|\mu\|_P(\{s \in S: \sup_n \{|f_n(s)|\} > \lambda\})$$

$$= \|\mu\|_P(\lim A_n)$$

$$= \lim_n \|\mu\|_P(A_n)$$

$$\le \frac{1}{\lambda} \sup \{\int |f_\tau| d|x^*\mu| : \tau \in T,\ x^* \in P\} .$$

Proposition 4.5. If $(f_n, \Sigma_n, n \in Z)$ is an amart then

$$\sup \{|\int f_\tau d(x^*\mu)| : x^* \le P,\ \tau \in T\} < \infty$$

for each continuous semi-norm P.

Proof. Since the net $\{\int f_\tau d\mu: \tau \in T\}$ converges in X it is Cauchy; consequently for each continuous semi-norm P there exists a positive integer N such that

$$\sup \{|x^*(\int f_\tau d\mu - \int f_N d\mu)| : x^* \le P\} < 1$$

for all $\tau \ge N$. Therefore for any bounded stopping time τ

$$|x^*(\int f_{\tau\wedge N} d\mu)| = |\int f_{\tau\wedge N} d(x^*\mu)| \le \int \max_{1\le n\le N} |f_n| d|x^*\mu|$$

and

$$|x^*(\int f_{\tau\vee N} d\mu - \int f_N d\mu)| \le 1$$

for all $x^* \le P$. As a result, we have

$$\begin{aligned}
&\sup\{|x^*\textstyle\int f_\tau d\mu| : x^* \le P\} \\
&= \sup\{|x^*(\textstyle\int f_{\tau\wedge N}d\mu + \int f_{\tau\vee N}d\mu - \int f_N d\mu)| : x^* \le P\} \\
&\le \sup\{|x^*(\textstyle\int f_{\tau\wedge N}d\mu)| : x^* \le P\} \\
&+ \sup\{|x^*(f_{\tau\vee N}d\mu - \textstyle\int f_N d\mu) : x^* \le P\} \\
&\le \sup\{\textstyle\int \max\limits_{1\le n\le N} |f_n| d|x^*\mu| : x^* \le P\} + 1 \\
&< \infty .
\end{aligned}$$

Q.E.D.

Theorem 4.6. (Nikodým) [9, p. 309] If M is a set in Ca (S,Σ), the space of countably additive set functions defined on Σ, and if for each $E \in \Sigma$ there is an $N(E) < \infty$ such that $|\mu(E)| < N(E)$ for all $\mu \in M$, then there is a number $N < \infty$ such that $|\mu(E)| < N$ for all $\mu \in M$ and $E \in \Sigma$.

The following results extend the theory of amarts found in Edgar and Sucheston [10] to the vector-valued case.

Lemma 4.7. Let (f_n) and (g_n) be sequences adapted to an increasing sequence (Σ_n) of sub-σ-fields of Σ such that for each continuous semi-norm P

$$\sup\{\textstyle\int |g_n| d|x^*\mu| : x^* \le P,\ n \in Z\} < \infty$$

and

$$\sup\{\textstyle\int |f_n| d|x^*\mu| : x^* \le P,\ n \in Z\} < \infty .$$

Then

(a) if $\sup\{\int f_\tau d|x^*\mu| : x^* \le P,\ \tau \in T\} < \infty \quad (> -\infty)$

and $\sup \{\int g_\tau d|x^*\mu| : x^* \leq P,\ \tau \in T\} < \infty \quad (> -\infty)$,

then $\sup \{\int (f_\tau \vee g_\tau) d|x^*\mu| : x^* \leq P,\ \tau \in T\} < \infty$

$(\inf \{\int (f_\tau \wedge g_\tau) d|x^*\mu| : x^* \leq P,\ \tau \in T\} > -\infty)$;

(b) if V is a sequentially complete space and if $(f_n, \Sigma_n : n \in Z)$, $(g_n, \Sigma_n : n \in Z)$ are amarts, then $(f_n \vee g_n, \Sigma_n : n \in Z)$ and $(f_n \wedge g_n, \Sigma_n : n \in Z)$ are also amarts.

Proof. For $\tau \in T$ choose $N \in Z$ such that $N \geq \tau$ and define $\sigma, \sigma' \in T$ as follows

$$\begin{aligned} \sigma(s) &= \tau(s) && \text{if } f_\tau(s) \geq 0 \\ &= N && \text{if } f_\tau(s) < 0 \\ \sigma'(s) &= \tau(s) && \text{if } g_\tau(s) \geq 0 \\ &= N && \text{if } g_\tau(s) < 0 . \end{aligned}$$

Then for any semi-norm P and $x^* \leq P$

$$\begin{aligned} \int (f_\tau \vee g_\tau) d|x^*\mu| &\leq \int_{\{f_\tau \geq 0\}} f_\tau d|x^*\mu| + \int_{\{g_\tau \geq 0\}} g_\tau d|x^*\mu| \\ &\leq \int f_\sigma d|x^*\mu| - \int_{\{f_\tau < 0\}} f_N d|x^*\mu| \\ &\quad + \int g_{\sigma'} d|x^*\mu| - \int_{\{g_\tau < 0\}} g_N d|x^*\mu| \\ &\leq \sup \{\int f_\tau d|x^*\mu| : \tau \in T\} \\ &\quad + \sup \{\int |f_n| d|x^*\mu| : n \in Z\} \\ &\quad + \sup \{\int g_\tau d|x^*\mu| : \tau \in T\} \\ &\quad + \sup \{\int |g_n| d|x^*\mu| : n \in Z\} \\ &= K < \infty . \end{aligned}$$

Consequently for any $\tau \in T$

$$\sup \{\int (f_\tau \vee g_\tau) d|x^*\mu| : x^* \leq P\} \leq K .$$

Therefore $\sup \{\int (f_\tau \vee g_\tau) d|x^*\mu| : x^* \leq P,\ \tau \in T\} < \infty$.

The companion statement

$$\inf \{f(f_\tau \vee g_\tau) d|x^*\mu| : x^* \leq P,\ \tau \in T\} > -\infty$$

is proved similarly.

To establish (b) we will prove that $(f_n \vee g_n, \Sigma_n : n \in Z)$ is an amart. The proof that $(f_n \wedge g_n, \Sigma_n : n \in Z)$ is an amart is almost identical.

Let P be a continuous semi-norm and define $h_n = f_n \vee g_n$ for all $n \in Z$. By Proposition 4.5

$$\sup \{\int f_\tau d(x^*\mu) : x^* \leq P,\ \tau \in T\} < \infty$$

and

$$\sup \{\int g_\tau d(x^*\mu) : x^* \leq P,\ \tau \in T\} < \infty .$$

Define $\Phi_\tau(\cdot) = \int_{(\cdot)} f_\tau d\mu$. Then by theorem 4.3 Φ_τ is a countably additive measure and

$$\begin{aligned} \sup \{\int f_\tau d|x^*\mu| : x^* \leq P,\ \tau \in T\} &\leq \sup \{\int |f_\tau| d|x^*\mu| : x^* \leq P,\ \tau \in T\} \\ &= \sup \{\| \Phi_\tau \|_P(S) : \tau \in T\} \\ &\leq 4 \cdot \sup \{|\int_E f_\tau d(x^*\mu)| : x^* \leq P, \tau \in T, E \in \Sigma\}. \end{aligned}$$

For each $E \in \Sigma$, the restriction of the sequence (f_n) to E is an amart for $(\Sigma_n \cap E : n \in Z)$, where $\Sigma_n \cap E$ denotes the

σ-field consisting of all intersections of elements of Σ_n with E. The restriction of (f_n) is clearly adapted to $(\Sigma_n \cap E: n \in Z)$. Since

$$\sup \{\int_S |f_n| d|x^*\mu|: x^* \le P\} < \infty$$

for any continuous semi-norm P it is clear that

$$\sup \{\int_E |f_n| d|x^*\mu|: x^* \le P\} < \infty .$$

To see that $\lim_{\tau \in T} \int_E f_\tau d\mu$ exists in V let $\varepsilon > 0$ be given and P be a continuous semi-norm on V. Then since $\lim_{\tau \in T} \int_S f_\tau d\mu$ exists, there exists an $N \in Z$ such that if $\tau, \sigma \ge N$ then

$$\sup \{|\int f_\tau d(x^*\mu) - \int f_\sigma d(x^*\mu)|: x^* \le P\} < \varepsilon .$$

Define $\tau_1, \sigma_1 \ge N$ by

$$\begin{aligned} \tau_1(s) &= \tau(s) && \text{if } s \in E \\ &= N && \text{if } s \notin E \\ \sigma_1(s) &= \sigma(s) && \text{if } s \in E \\ &= N && \text{if } s \notin E . \end{aligned}$$

Then

$$\begin{aligned} &\sup \{|\int_E f_\tau d(x^*\mu) - \int_E f_\sigma d(x^*\mu)|: x^* \le P\} \\ &= \sup \{|\int_S f_{\tau_1} d(x^*\mu) - \int_S f_{\sigma_1} d(x^*\mu)|: x^* \le P\} \\ &< \varepsilon . \end{aligned}$$

Therefore $(\int_E f_\tau d\mu \colon \tau \in T)$ is Cauchy in V and hence, since V is sequentially complete, the subsequence $(\int_E f_n d\mu \colon n \in Z)$ converges in V. From this it follows that $(\int_E f_\tau d\mu \colon \tau \in T)$ converges in V. This completes the verification that for each $E \in \Sigma$ the restriction of (f_n) to E is an amart for $(\Sigma_n \cap E \colon n \in Z)$.

For each $E \in \Sigma$, by restricting the elements of the sequence (f_n) to E it is seen that Proposition 4.5 implies that

$$\sup \{|\int_E f_\tau d(x^*\mu)| \colon x^* \leq P,\ \tau \in T\} < \infty$$

for each continuous semi-norm P. Hence by applying Theorem 4.6 (the Nikodým Boundedness Theorem) to the family

$$\{\int_{(\cdot)} f_\tau d(x^*\mu) \colon x^* \leq P,\ \tau \in T\}$$

of scalar-valued measures it follows that

$$\sup \{|\int_E f_\tau d(x^*\mu)| \colon x^* \leq P,\ \tau \in T,\ E \in \Sigma\} < \infty\ ;$$

consequently

$$\begin{aligned} &\sup \{\int_S f_\tau d|x^*\mu| \colon x^* \leq P,\ \tau \in T\} \\ &\quad \leq 4 \sup \{|\int_E f_\tau d(x^*\mu)| \colon x^* \leq P,\ \tau \in T,\ E \in \Sigma\} \\ &\quad < \infty\ . \end{aligned}$$

By an identical argument

$$\sup \{\int_S g_\tau d|x^*\mu| \colon x^* \leq P,\ \tau < T\} < \infty\ .$$

For any $\tau \in T$ denote $f_\tau \vee g_\tau$ by h_τ. By part (a)

$$\sup \{\int h_\tau d|x^*\mu| : x^* \le P, \tau \in T\} < \infty .$$

Since both $(\int f_\tau d\mu : \tau \in T)$ and $(\int g_\tau d\mu : \tau \in T)$ are convergent and hence Cauchy nets in V, for any continuous semi-norm P and any $\varepsilon > 0$ there is a $\tau_o \in T$ such that if $\sigma, \tau \ge \tau_o$ then (*) $P(\int f_\sigma d\mu - \int f_\tau d\mu) < \varepsilon$ and $P(\int g_\sigma d\mu - \int g_\tau d\mu) < \varepsilon$.

Since

$$\sup \{\int h_\tau d|x^*\mu| : x^* \le P, \tau < T\} < \infty$$

it is possible to choose $\tau_1 \ge \tau_o$ such that if $\sigma \ge \tau_o$ then

$$\sup \{\int h_\sigma d|x^*\mu| : x^* \le P\} \le \sup \{\int h_{\tau_1} d|x^*\mu| : x^* \le P\} + \varepsilon .$$

For any bounded stopping time $\sigma \ge \tau_1$, define

$$A = \{s : f_{\tau_1}(s) < g_{\tau_1}(s)\}$$

and define $\sigma_1 \in T$ by

$$\begin{aligned} \sigma_1(s) &= \tau_1(s) \quad \text{if} \quad s \in A \\ &= \sigma(s) \quad \text{if} \quad s \notin A . \end{aligned}$$

Then

$$\int f_{\tau_1} d(x^*\mu) = \int_{A^c} h_{\tau_1} d(x^*\mu) + \int_A f_{\tau_1} d(x^*\mu)$$

$$\int f_{\sigma_1} d(x^*\mu) = \int_{A^c} f_\sigma d(x^*\mu) + \int_A f_{\tau_1} d(x^*\mu) .$$

Subtracting the second equality above from the first we have

$$\int_{A^c} h_{\tau_1} d(x^*\mu) = \int_{A^c} f_\sigma d(x^*\mu) + \int f_{\tau_1} d(x^*\mu) - \int f_{\sigma_1} d(x^*\mu)$$

for each $x^* \le P$. Using (*) we have

$$\begin{aligned}\sup\{|\int_{A^c} h_{\tau_1} d(x^*\mu)|: x^* \le P\} &\le \sup\{|\int_{A^c} f_\sigma d(x^*\mu)|: x^* \le P\} \\ &+ \sup\{|\int f_{\tau_1} d(x^*\mu) - \int f_{\sigma_1} d(x^*\mu)|: x^* \le P\} \\ &\le \sup\{|\int_{A^c} h_\sigma d(x^*\mu)|: x^* \le P\} + \varepsilon \quad .\end{aligned}$$

Similarly

$$\sup\{|\int_A h_{\tau_1} d(x^*\mu)|: x^* \le P\} \le \sup\{|\int_A h_\sigma d(x^*\mu)|: x^* \le P\} + \varepsilon.$$

Therefore

$$\sup\{|\int h_{\tau_1} d(x^*\mu)|: x^* \le P\} \le \sup\{|\int h_\sigma d(x^*\mu)|: x^* \le P\} + 2\,\varepsilon.$$

This implies that

$$\sup\{|\int h_\sigma d(x^*\mu) - \int h_{\tau_1} d(x^*\mu)|: x^* \le P\} \le 2\,\varepsilon\ ,$$

and therefore $(\int h_\tau d\mu: \tau \in T)$ is Cauchy in X. Consequently, since X is sequentially complete, the subsequence $(\int h_n d\mu: n \in Z)$ is Cauchy and thus convergent in V. It follows that the net $(\int h_\tau d\mu: \tau \in T)$ converges in V and hence $(h_n, \Sigma_n, n \in Z) = (f_n \vee g_n, \Sigma_n, n \in Z)$ is an amart.

Q.E.D.

An amart $(f_n, \Sigma_n, n \in Z)$ is L'-bounded if

$$\sup\{|\int f_n| d|x^*\mu|: x^* \le P, n \in Z\} < \infty$$

for each continuous semi-norm P.

Proposition 4.8. Let the sequence (f_n) be an amart for $(\Sigma_n: n \in Z)$. If $(f_n: n \in Z)$ is L^1-bounded, then:

(a) $(f_n : n \in Z)$, $(f_n^+: n \in Z)$, $(f_n^-: n \in Z)$, and for $\lambda \geq 0$, $(-\lambda \vee f_n \wedge \lambda: n \in Z)$, are also L^1-bounded amarts for $(\Sigma_n, n \in Z)$.

(b) $\sup \{|\int |f_\tau| d(x^*\mu)| : \tau \in T, x^* \leq P\} < \infty$.

(c) $\sup \{|f_n| : n \in Z\} < \infty$ μ-almost everywhere.

Proof. (a) Since $(f_n: n \in Z)$ and $(-f_n: n \in Z)$ are amarts it follows from Lemma 4.7 that $(|f_n|: n \in Z) = (f_n \vee (-f_n): n \in Z)$ is an amart. Also, since a sequence of constants is an amart, $(f_n^+: n \in Z) = (f_n \vee 0: n \in Z)$, $(f_n^-: n \in Z) = (-(f_n \wedge 0): n \in Z)$ and $(-\lambda \vee f_n \wedge \lambda: n \in Z)$ are L^1-bounded amarts.

(b) By part (a), $(|f_n|)$ is an L^1bounded amart so by Proposition 4.5,

$$\sup \{|\int |f_\tau| d(x^*\mu)| : x^* \leq P, \tau \in T\} < \infty .$$

Part (c) follows from part (b) and Lemma 4.4.

Proposition 4.9. If Y is a random variable such that for each $s \in S$, $Y(s)$ is a cluster point of $(f_n(s): n \in Z)$, then for every integer K there exists a $\tau_K \in T$ with $\tau_{K+1} \geq \tau_K$ and $\tau_K \geq K$ such that $\lim_{K\to\infty} f_{\tau_K}(s) = Y(s)$ for almost all $s \in S$.

Proof. Let P be a continuous semi-norm and choose $\varepsilon > 0$. Since Y is Σ-measurable and $\Sigma = \sigma\{\cup_{n\geq 1} \Sigma_n\}$ there is an integer N_1 and a random variable Y_1 which is Σ_{N_1}-measurable such that

$$\|\mu\|_p\{|Y - Y_1| < {}^{\varepsilon}/4\} > \|\mu\|_p(S) - {}^{\varepsilon}/4 .$$

Since $Y(s)$ is a cluster point of $(f_n(s): n \in Z)$ for all $s \in S$, it follows that

$$\|\mu\|_p\{|f_n - Y| < {}^{\varepsilon}/4 \text{ for some } n \geq N_1\} \geq \|\mu\|_p(S) - {}^{\varepsilon}/4 .$$

Thus

$$\|\mu\|_p\{|f_n - Y_1| < {}^{\varepsilon}/2 \text{ for some } n \geq N_1\} \geq \|\mu\|_p(S) - {}^{\varepsilon}/2 .$$

We can therefore find an integer $N_2 \geq N_1 \vee 2$ such that if A_1 is defined by

$$A_1 = \{|f_n - Y_1| < {}^{\varepsilon}/2 \text{ for some } n,\ N_1 \leq n \leq N_2\},$$

then $\|\mu\|_p(A_1) \geq \|\mu\|_p(S) - {}^{\varepsilon}/2$. Define $\tau_1: S \rightarrow Z$ by

$$\begin{aligned} \tau_1(s) &= \min\{n: N_1 \leq n \leq N_2 \text{ and } |f_n(s) - Y_1(s)| < {}^{\varepsilon}/2\} && \text{if } s \in A_1 \\ &= N_2 && \text{if } s \notin A_1. \end{aligned}$$

Then $\tau_1 \in T$ and $\|\mu\|_p\{|f_{\tau_1} - Y| < \varepsilon\} > \|\mu\|_p(S) - \varepsilon$.

Now define (τ_K) inductively using $\varepsilon = {}^1/K$. That is, suppose τ_K has been defined as above by

$$\begin{aligned} \tau_K(s) &= \min\{n: N_K \leq n \leq N_{K+1} \text{ and } |f_n(s) - Y_K(s)| < {}^1/2K\} && \text{if } s \in A_K \\ &= N_{K+1} && \text{if } s \notin A_K. \end{aligned}$$

Then following the same argument as above there is an integer $N_{K+2} \geq N_{K+1} \geq (K+2)$ such that

$$\| \mu \|_P (A_{K+1}) \quad 1 - \frac{1}{2(K+1)} ,$$

where

$$A_{K+1} = \{ |f_n - Y_{K+1}| < \frac{1}{2(K+1)} \text{ for some } n, \ N_{K+1} \leq n \leq N_{K+2} \}.$$

Define τ_{K+1} by

$$\begin{aligned} \tau_{K+1}(s) &= \min \{ n: N_{K+1} \leq n \leq N_{K+2} \text{ and } |f_n(s) - Y_{K+1}(S)| \\ &\quad < \frac{1}{2(K+1)} \} && \text{if } s \in A_{K+1} \\ &= N_{K+2} && \text{if } s \notin A \end{aligned}$$

and the proposition follows.

Q.E.D.

The proof of the following lemma is essentially the same as the proof in Edgar and Sucheston [10] for the scalar-valued case.

Lemma 4.10. Let (f_n) be a sequence of random variables adapted to $(\Sigma_n: n \in Z)$ and suppose there is an integrable random variable g such that $|f_n| \leq g$ for all $n \in Z$. Then if V is sequentially complete, the following are equivalent:

(a) $(f_n, \Sigma_n: n \in Z)$ is an amart;

(b) the sequence (f_n) converges almost everywhere.

Proof. Suppose that $(f_n, \Sigma_n: n \in Z)$ is an amart. Define f^* and f_* by

$$f^*(s) = \overline{\lim_n} f_n(s), \ f_*(s) = \underline{\lim_n} f_n(s) .$$

Then by Proposition 4.9 there are increasing sequences $(\tau_n: n \in Z)$ and $(\sigma_n: n \in Z)$ of bounded stopping times such that $\lim_n f_{\tau_n}(s) = f^*(s)$ and $\lim_n f_{\sigma_n}(s) = f_*(s)$ for almost all $s \in S$. Since g is μ-integrable by assumption and

$$(f^* - f_*) \leq 2g \ ,$$

applying Theorem 4.3 we have

$$\int(f^* - f_*)d\mu = \lim_n \int(f_{\tau_n} - f_{\sigma_n})d\mu = 0 \ .$$

Therefore $f^* = f_*$ almost everywhere and thus the sequence (f_n) converges a.e. .

Conversely, suppose (f_n) converges a.e. to f. If $(\tau_n: n \in Z)$ is an increasing sequence of bounded stopping times, then $f_{\tau_n} \rightarrow f$ a.e., and since $|f_{\tau_n}| < g$, which is integrable by assumption, Theorem 4.3 applies and we have

$$\lim_n (\int f_{\tau_n} d\mu) = \int f d\mu \ .$$

Therefore

$$\lim_\tau (\int f_{\tau_n} d\mu) = \int f d\mu \ ,$$

concluding the proof that $(f_n, \Sigma_n: n \in Z)$ is an amart.

Q.E.D.

Lemma 4.11. Let $(f_n, \Sigma_n: n \in Z)$ be an amart and assume that for each continuous semi-norm P

$$\sup \{\int |f_n| d|x^*\mu| : n \in Z, \ x^* \leq P\} < \infty \ .$$

Then the sequence (f_n) converges μ-a.e. if V is sequentially complete.

Proof. By Proposition 4.8 $\sup \{|f_n|: n \in Z\} < \infty$ μ-a.e. therefore

$$\lim_{\lambda\to\infty} \{\sup_n |f_n| > \lambda\} = E ,$$

where E is a μ-null set. By Theorem 4.2

$$\lim_{\lambda\to\infty} \| \mu \|_P\{\sup_n |f_n| > \lambda\} = \| \mu \|_P(E) = 0 .$$

Since by Proposition 4.8 the sequence $(-\lambda \vee f_n \vee \lambda: n \in Z)$ is an amart for any $\lambda \geq 0$, Lemma 4.10 implies that $(-\lambda \vee f_n \wedge \lambda: n \in Z)$ converges μ-a.e. . For any $\varepsilon > 0$ and continuous semi-norm P choose λ such that

$$\| \mu \|_P\{\sup_n |f_n| > \lambda\} < \varepsilon .$$

Since

$$\{f_n \neq (-\lambda \vee f_n \wedge \lambda) \text{ for some } n\} = \{\sup_n |f_n| > \lambda\},$$

it follows that the set of all $s \in S$ for which the sequence $(f_n(s))$ does not converge has P-semi-variation less than ε. Consequently (f_n) converges μ-a.e. .

Q.E.D.

The next lemma constitutes an "optional sampling theorem" for amarts. The proof is modeled after a similar theorem for scalar-valued amarts in [10].

Lemma 4.12. Let $(f_n, \Sigma_n: n \in Z)$ be an amart and let $(\tau_n: n \in Z)$ be a non-decreasing sequence of bounded stopping times for $(\Sigma_n : n \in Z)$. Define $g_n = f_{\tau_n}$ and

$$\Sigma'_K = \Sigma_{\tau_K} = A \in \Sigma : A \cap \{\tau_K = n\} \in \Sigma_n ,\ n \in Z\} \quad .$$

Then $(g_n, \Sigma'_n : n \in Z)$ is an amart if V is sequentially complete.

Proof. Since for each $K \in Z$

$$\{g_K \leq t\} = \bigcup_{n \geq 1} \{\{f_n \leq t\} \cap \{\tau_K = n\}\} \in \Sigma$$

and

$$\{g_K \leq t\} \cap \{\tau_K = n\} = \{f_n \leq t\} \in \Sigma_n \quad ,$$

it follows that $\{g_K \leq t\} \in \Sigma'_K$ for any $t \in \Re$. Therefore g_K is Σ'_K measurable for all $K \in Z$.

If σ is a stopping time for $(\Sigma'_n : n \in Z)$ then τ_σ is a stopping time for $(\Sigma_n : n \in Z)$. This follows from the observation that if σ is a stopping time for $(\Sigma'_n : n \in Z)$ then $\{\sigma = K\} \in \Sigma'_K$, which implies that

$$\{\{\tau_K = n\} \cap \{\sigma = K\}\} \in \Sigma_n \text{ for } K, n \in Z \ ,$$

and thus

$$\{\tau_\sigma = n\} = \bigcup_{K \geq 1} \{\{\tau_K = n\} \cap \{\sigma = K\}\}$$

is in Σ_n for all $n \in Z$.

For any $\varepsilon > 0$ and any continuous semi-norm P, choose N so that $P(\int f_\tau d\mu - \int f_{\tau'} d\mu) < \varepsilon/2$ for all bounded stopping times $\tau, \tau' \geq N$. Define $\tau_\infty = \lim_K \tau_K$. Then $f_{t_K \wedge N} \longrightarrow f_{\tau_\infty \wedge N}$

$K \longrightarrow \infty$ and

$$\sup \{\int \sup_K |f_{\tau_K \wedge N}| d|x^*\mu| : x^* \le P\} \le$$

$$\sup \{\int (|f_1| \vee \ldots \vee |f_N|) d|x^*\mu| : x^* \le P\} \le \infty .$$

Therefore by Theorem 4.3 $(\int f_{\tau_K \wedge N} d\mu : K \in Z)$ converges in V. Consequently $(f_{\tau_K \wedge N}, \Sigma'_K : K \in Z)$ is an amart.

Since $(\int f_{\tau_K \wedge N} d\mu : K \in Z)$ converges in V there is an $M \in Z$ such that if σ, σ' are two bounded stopping times for $(\Sigma'_K : K \in Z)$ with $\sigma, \sigma' \ge M$ then

$$P(\int f_{\tau_\sigma \wedge N} d\mu - \int f_{\tau_{\sigma'} \wedge N} d\mu) < \varepsilon/2 .$$

Thus for $\sigma, \sigma' \ge M$ we have

$$P(\int g_\sigma d\mu - \int g_{\sigma'} d\mu) = P(\int f_{\tau_\sigma} d\mu - \int f_{\tau_{\sigma'}} d\mu) =$$

$$P(\int f_{\tau_\sigma \vee N} d\mu + \int f_{\tau_\sigma \wedge N} d\mu - \int f_{\tau_{\sigma'} \vee N} d\mu - \int f_{\tau_{\sigma'} \wedge N} d\mu) \le$$

$$P(\int f_{\tau_\sigma \vee N} d\mu - \int f_{\tau_{\sigma'} \vee N} d\mu + P(\int f_{\tau_\sigma \wedge N} d\mu - \int f_{\tau_{\sigma'} \wedge N} d\mu) \le \varepsilon .$$

We have shown that for any $\varepsilon > 0$ and continuous semi-norm P there exists an integer M, such that if σ, σ' are two bounded stopping times with $\sigma, \sigma' \ge M$ then $P(\int g_\sigma d\mu - \int g_{\sigma'} d\mu) < \varepsilon$. Therefore $(g_n, \Sigma'_n : n \in Z)$ is an amart.

Q.E.D.

Proposition 4.13. If $(f_n, \Sigma_n : n \in Z)$ is an amart and σ is a stopping time then $(f_{n \wedge \sigma}, \Sigma_n : n \in Z)$ is an amart if V is sequentially complete.

Proof. Let $\tau_n = n \wedge \sigma$ and apply Lemma 4.12.

Q.E.D.

The next convergence theorem is for amarts which are not necessarily L^1-bounded. The case when $(f_n: n \in Z)$ is a real-valued martingale and $\tau_K = K$ is due to Doob [8, p. 320]. The case when $(f_n, \Sigma_n, n \in Z)$ is a real-valued amart is due to Edgar and Sucheston [10, p. 204].

Proposition 4.14. If $(f_n, \Sigma_n: n \in Z)$ is an amart and $(\tau_K: K \in Z)$ an increasing sequence of bounded stopping times with $\tau_K \geq K$, and if for each continuous semi-norm P

$$\sup \{\int (\sup_{K \in Z} |f_{\tau_K} - f_{\tau_{K-1}}|) d|x^*\mu| : x^* \leq p\} < \infty ,$$

then (f_n) converges a.e. on the set $G = \{\sup_n |f_n|\} < \infty$.

Proof. For $\lambda > 0$ define σ and σ' as follows: if K is the smallest integer such that $|f_K| > \lambda$, then $\sigma' = K$ and $\sigma = \tau_K$; if $|f_K| < \lambda$ for all K then $\sigma' = \sigma = \infty$. Define a sequence of stopping times $(\tau'_n: n \in Z)$ by

$$\begin{aligned} \tau'_n(s) &= n \quad \text{if } \sigma'(s) > n \\ &= \sigma(s) \quad \text{if } \sigma'(s) \leq n \quad . \end{aligned}$$

If $\sigma' \leq n$, then $\sigma = \tau_K \leq \tau_n$, so τ'_n is clearly bounded. By Lemma 4.12 $(f_{\tau'_n}, \Sigma_n, n \in Z)$ is an amart. If $\sigma' > n$, then

$$|f_{\tau'_n}| = |f_n| \leq \lambda$$

If $\sigma' \leq n$, then

$$|f_{\tau'_n}| = |f_\sigma| = |f_{\tau_K}| ,$$

where K is the smallest integer such that $|f_K| > \lambda$. Note that if $\sigma' \le n$, then $|f_{K-1}| \le \lambda$ and thus

$$|f_{\tau'_n}| - \lambda = |f_{\tau_K}| - \lambda \le |f_{\tau_K}| - |f_{K-1}| \le |f_{\tau_K} - f_{K-1}| \quad .$$

It follows that for all $n \in Z$

$$|f_{\tau'_n}| \le \lambda + \sup_K |f_{\tau_K} - f_{K-1}| \ .$$

Therefore for each continuous semi-norm P

$$\sup \{\int |f_{\tau'_n}| d|x^*\mu| : x^* \le P,\ n \in Z\}$$

$$\le \sup \{\int |\lambda + \sup |f_{\tau_K} - f_{K-1}|| d|x^*\mu| : x^* \le P\} < \infty \ .$$

Thus by Lemma 1.11 the sequence $(f_{\tau'_n})$ converges a.e. . Since $f_{\tau'_n} = f_n$ on the set $G_\lambda = \{\sup_K |f_K| \le \lambda\}$, (f_n) converges a.e. on $\bigcup_{\lambda \in Z} G_\lambda = \{\sup_K |f_K| < \infty\} \equiv G$.

Q.E.D.

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BIOGRAPHICAL SKETCH

Robert Wayne Smith was born in Murphyshore, Illinois, on November 30, 1947. He graduated from Murphyshore High School in 1965 and received his B. S. in Corporation Finance from Southern Illinois University in 1970. He spent two years as a peace corps volunteer in Honduras. He then returned to Southern Illinois University, where he received an M. S. in Mathematics in 1974. After a year of teaching English in Uruguay and traveling throughout South America, he returned to the United States to resume graduate study in mathematics at the University of Florida in 1975.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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